

SUPPLEMENT TO “LOWER BOUNDS FOR THE QUANTUM PERMANENT”

SUVRIT SRA*

ABSTRACT. This supplement collects material removed from the paper “Lower bounds for the quantum permanent” because it is superseded by the results there or consists of certificate data: the earlier finite certificate releases and their single-formula envelopes, the exact constants of the dimension-twenty-four certificate, three explicit recursive logarithmic bounds, and the three-shape feasible profile that first limited the recursive relaxation. Sections S4–S6 hold results outside the paper’s main line: the proof that $3e^2 - 7$ is the optimal logarithmic coefficient of the recursive relaxation, with two obstructions for weaker relaxations; bounds from the structure of the map that support the conjectured order of the minimum; and the contraction, concentration, and minimizer results. Numbers without the prefix S refer to the paper; all notation is that of the paper, and the supporting files hold the dual vectors and rational checks.

NOTATION FROM THE PAPER

For $H \in \mathfrak{B}_n$, the orbital weights w_λ and scores r_λ are those of (2.3), and $\mathcal{L}_n$ is the signed coefficient functional there. For a seed (k, γ) ,

$$h_n(K) = -\frac{n-1-K}{n(n-1)+K}, \quad K(k, \gamma) = \frac{(k-1)(1+k\gamma)}{1-\gamma},$$

as in (3.1) and (3.2). The transport certificate (A.5) writes $\Phi_n(H) - h_n(K)$ as $tJ_{n,k,\gamma}$ plus a parity multiple of R_n plus $\sum_\lambda c_\lambda w_\lambda$ with residuals c_λ ; (A.6) and (A.7) are its shifted residual polynomials $P_{j,s}$ and unselected-shape test. The finite program is (3.4), with the rational model (A.8)–(A.9), and $\mathcal{D}_k$, $\mathcal{T}_k$ are the radial operators of the paper’s Section 2.1.

S1. FINITE CERTIFICATES AND ENVELOPES

The paper’s Table 1 lists the strongest certified seeds. This section records the earlier certificate releases, the strengthened dimension-eighteen seed, the exact constants of the dimension-twenty-four certificate (Proposition A.3 of the paper), and the single-formula envelopes that preceded the bound $-3n/(3n^2 + 20n + 103)$ of Theorem 1.2. Table 1 below is the frozen first release; the values for $n = 8$ and $n = 18$ were later strengthened as described in Sections S1.2 and S1.3.

S1.1. The first certificate release.

Proposition S1.1 (Finite certificates). *The exact dual vectors supplied with this paper certify the bounds in Table 1. In dimensions four through seven, matching rational primal vectors certify the optimum of the stated recursive relaxation. In dimensions eight through eighteen, the values are feasible dual bounds without an optimum claim. The seeds in dimensions seventeen and eighteen pass the transport residual test from dimensions eighteen and nineteen, respectively.*

Proof. The coefficient formulas (A.4), (A.8), and (A.9) specify the rational model. The supplied verifier rebuilds it, checks every entry of $c - A^\top y$, and computes $b^\top y$ exactly. For dimensions four through seven it also checks $Ax = b$, $x \geq 0$, and equality of the primal and dual objectives. In dimensions eight through eighteen, Table 1 rounds the exact dual objective downward to the displayed rational γ_n ; the verifier checks $b^\top y \geq \gamma_n$ before using that seed.

*TU Munich, Dept. of Mathematics, School of CIT; Garching, Germany. Email: s.sra@tum.de.

Table 1: Certified lower bounds $\gamma_n \leq \mu_n$. Every terminating decimal is an exact rational number. The two longer fractions are given in (S1.8).

n	γ_n	status	n	γ_n	status
4	$-3/62$	LP optimum	12	-0.04338172	dual
5	$-77957/1378367$	LP optimum	13	-0.0414257677	dual
6	see (S1.8)	LP optimum	14	-0.0394550826	dual
7	see (S1.8)	LP optimum	15	-0.0379279304	dual
8	-0.05274023	dual	16	-0.0365074575	dual
9	-0.05064282	dual	17	-0.0352435158	dual
10	-0.04791679	dual	18	-0.0342921802	dual
11	-0.04573151	dual			

For the dimension-seventeen and dimension-eighteen seeds, the transport verifier checks (A.7) and the shifted coefficients in (A.6), starting at $n_0 = k + 1$. There are respectively 73 and 82 pairs (j, s) , including the two identically zero polynomials. The explicit parity residual in the proof of Theorem 3.1 covers the remaining boundary coefficient. The dual vectors, the rational model builders and the verifiers accompany this paper as supporting files. A floating-point optimizer only proposed the dual vectors; rational reconstruction and exact feasibility decide whether one certifies a bound. These are finite checks in rational arithmetic, rather than tests at finitely many target dimensions. $\square$

S1.2. A strengthened dimension-eighteen certificate. Retaining spectral moments through degree ten gives the stronger dimension-eighteen certificate

$$(S1.1) \quad \hat{\gamma}_{18} = -\frac{34110757033}{10^{12}}, \quad K(18, \hat{\gamma}_{18}) = K_{18}.$$

This is a lower certificate; no optimality over maps or over the finite relaxation is asserted.

The rational model is unchanged. A numerical solve retaining projection-spectrum equations through degree ten proposes a dual vector; exact rational reconstruction and verification against every model column give

$$b^\top y \geq -\frac{34110757033}{10^{12}}.$$

Thus (S1.1) is a valid seed. The dependencies are the already verified seeds in dimensions two through seventeen.

For this seed, all coefficients of the 82 polynomials in (A.6), shifted from $n_0 = 19$, are nonnegative. The unselected-shape test (A.7) and the parity correction also pass exactly. Hence its transported bound holds for every $n \geq 19$, and the seed supplies $n = 18$. The supporting files also include the ten comparisons for $8 \leq n \leq 17$.

S1.3. Finite certificates through dimension twenty-four. Each seed in Table 2 is a downward rounding of an exactly feasible dual for the full rational model. The dependencies include $\gamma_8 = -0.052740227999$ and the strengthened dimension-eighteen seed (S1.1). The verifier checks the entire dependency chain; no optimality of these dual values is asserted.

Proposition S1.2 (A dimension-twenty-four certificate). *Every $H \in \mathfrak{B}_{24}$ satisfies*

$$\Phi_{24}(H) \geq -\frac{649}{22351} = h_{24}(271/40).$$

The transported bound $\Phi_n(H) \geq h_n(271/40)$ holds for every $n \geq 24$.

Table 2: Further rational lower certificates. The terminating decimals are exact.

n	γ_n	$K(n, \gamma_n)$
19	-0.033049439588	558090971742/86087453299
20	-0.032188230644	564039362940/86015685887
21	-0.031356683435	1366038591460/206271336687

Proof. Fix $n = 24$, $k = 21$, $a = 12$, and $\gamma = \gamma_{21}$ from Table 2. Put

$$N = n(n-1), \quad K = K(k, \gamma), \quad h = h_n(K), \quad t = \frac{n^2 - 1}{(1 - \gamma)(N + K)}, \quad \eta = \frac{1}{n} + h,$$

$$v = \frac{a(n-a)}{N}, \quad \Delta = \frac{a(n-a)}{N + K} - \frac{1}{4}, \quad \alpha_0 = \frac{4nv}{n+1} = \frac{576}{575}.$$

The residuals in (A.5) are $c_\lambda = r_\lambda - h - tJ_\lambda - \eta R_\lambda$. They vanish at (1^n) , $(2, 1^{n-2})$, and (n) , and are positive at every other partition of n .

To constrain the two leading weights, define

$$C_{24} = \frac{49038552465408}{22017942472189}, \quad \psi(0) = 1, \quad \psi(\pm 1) = -1, \quad \psi(z) = C_{24} \quad (|z| \geq 2),$$

and let

$$b_\lambda = \binom{n}{a}^{-1} \sum_{|I|=a} \psi \left(\sum_{i \in I} \lambda_i - a \right), \quad B(H) = \sum_{\lambda \vdash n} b_\lambda w_\lambda.$$

For a spectral generator $u + \text{Bin}(b, p)$ with $u + bp = a$, the inequality $\mathbb{E}\psi(u + \text{Bin}(b, p) - a) \geq 0$ is a finite rational inequality. It holds for all 145 generators of the rank-twelve spectral hull, so $B(H) \geq 0$. Equality at $(u, b, p) = (0, 13, 12/13)$ fixes the displayed value of C_{24} . The two leading coefficients are $b_{(1^n)} = 1$ and $b_{(2, 1^{n-2})} = 1 - 4v$.

Define

$$d_\lambda = b_\lambda - 1 + \alpha_0(1 - r_\lambda), \quad d_o = d_{(n)}, \quad M = \max_{c_\lambda > 0} \frac{d_\lambda + d_o R_\lambda}{c_\lambda}.$$

The coefficients d_λ vanish at the two leading shapes. Since $R_{(n)} = -1$, the adjusted coefficient also vanishes at (n) . The maximum occurs at $\lambda = (6, 1^{18})$ and equals

$$M = \frac{35977984552366996045762853848991272}{327848126478907158293371233405225}.$$

Set $s_\lambda = M c_\lambda - d_\lambda - d_o R_\lambda$. Every s_λ is nonnegative, and summing the coefficient identities gives

$$(S1.2) \quad (\alpha_0 + M)(\Phi_n(H) - \gamma^\sharp) = B(H) + MtJ_{n,k,\gamma} + (M\eta + d_o)R_n + \sum_{\lambda \vdash n} s_\lambda w_\lambda,$$

where $\gamma^\sharp = h + 4\Delta/(\alpha_0 + M)$. All multipliers on the right are nonnegative. The exact value is

$$\gamma^\sharp = -\frac{9167110563519855245922921656813}{315707850865223035909120949619832} > -\frac{649}{22351}.$$

The supporting verifier checks the 145 spectral inequalities and all 1575 coefficient identities in (S1.2). Thus this seed requires only finite rational checks.

For the rounded seed $-649/22351$, all 145 transport polynomials in (A.6) have nonnegative coefficients after the shift $n = 25 + x$. The unselected-shape test and the parity residual also pass exactly. This proves the transported bound for $n \geq 25$; the seed itself supplies $n = 24$. $\square$

S1.4. Envelopes from the dimension-twenty-four seed. A stronger single rational envelope is

$$(S1.3) \quad \Phi_n(H) \geq -\frac{n}{n^2 + (271/40)n + B_{24}} \quad (n \geq 8), \quad B_{24} = \frac{8830525252591}{263701139995}.$$

Indeed, $B_{24} = -8/\gamma_8 - 64 - 8(271/40)$, so equality holds at $n = 8$ in the finite comparison. The remaining comparisons through $n = 23$ hold exactly. For the tail, the numerator in (S1.9) has positive constant and linear coefficients because $B_{24} > 0$ and $(271/40)^2 + 2(271/40) - B_{24} > 0$.

S1.5. Earlier envelopes.

Proposition S1.3 (The dimension-eighteen envelope). *Every $H \in \mathfrak{B}_n$ satisfies*

$$(S1.4) \quad \Phi_n(H) \geq -\frac{3n}{3n^2 + 19n + 111} \quad (n \geq 8).$$

With h_n as in (3.1), define

$$(S1.5) \quad K_{18} = \frac{6562108347902}{1034110757033} \approx 6.34565331.$$

Then $\Phi_n(H) \geq h_n(K_{18})$ for every $n \geq 18$. These bounds are lower certificates; no equality or optimality over maps is asserted.

Proof. The certificate (S1.1) and its threshold test give $\Phi_n \geq h_n(K_{18})$ for all $n \geq 18$. Since $K_{18} > 19/3$, monotonicity gives $h_n(K_{18}) \geq h_n(19/3)$. For $A = 19/3$ and $B = 37$,

$$h_n(A) + \frac{n}{n^2 + An + B} = \frac{(142/9)n + 814/3}{(n(n-1) + A)(n^2 + An + B)} > 0.$$

This proves (S1.4) for $n \geq 18$. For $8 \leq n \leq 17$, the exact seeds in Table 1 exceed the right side of (S1.4); the ten rational comparisons are included in the certificate verification. $\square$

Theorem S1.4 (A smaller-coefficient bound). *Every $H \in \mathfrak{B}_n$ satisfies*

$$(S1.6) \quad \Phi_n(H) \geq -\frac{5n}{5n^2 + 31n + 190} \quad (n \geq 8).$$

With h_n as in (3.1), put

$$(S1.7) \quad K_{18}^{(0)} = \frac{32532964294}{5171460901} \approx 6.29086537.$$

Then $\Phi_n(H) \geq h_n(K_{18}^{(0)})$ for every $n \geq 18$. These bounds are lower certificates; no equality or optimality over maps is asserted.

Proof of Theorem S1.4. The dimension-eighteen certificate is $\gamma_{18} = -171460901/5000000000$. Its transport constant is exactly $K_{18}^{(0)}$ in (S1.7). Proposition S1.1 and the endpoint identity in (3.3) prove $\Phi_n \geq h_n(K_{18}^{(0)})$ for every $n \geq 18$.

Since $K_{18}^{(0)} > 31/5$, monotonicity in (3.3) gives $h_n(K_{18}^{(0)}) \geq h_n(31/5)$. With $A = 31/5$ and $B = 38$, cross-multiplication gives

$$h_n(A) + \frac{n}{n^2 + An + B} = \frac{(321/25)n + 1368/5}{(n(n-1) + A)(n^2 + An + B)} > 0.$$

This proves (S1.6) for $n \geq 18$. For each $8 \leq n \leq 17$, the exact entry in Table 1 is at least $-5n/(5n^2 + 31n + 190)$; the accompanying rational comparisons check these ten cases. $\square$

For dimension seventeen, the same argument gives $\Phi_n \geq h_n(K_{17})$ for $n \geq 17$, where

$$K_{17} = \frac{32068818512}{5176217579}.$$

S1.6. Further envelopes and dimension-free consequences. The two seed values with nonterminating decimal expansions in Table 1 are

$$(S1.8) \quad \begin{aligned} \gamma_6 &= -\frac{129358525188828749}{2325438043571344896}, \\ \gamma_7 &= -\frac{5408353173620987686795318309001825}{97775984492669811845528183241796197}. \end{aligned}$$

To form a single rational bound from a transported seed, take A, B with $n^2 + An + B > 0$. Cross-multiplication gives

$$(S1.9) \quad h_n(K) + \frac{n}{n^2 + An + B} = \frac{(K - A)n^2 + [K + A(K + 1) - B]n + B(K + 1)}{(n(n - 1) + K)(n^2 + An + B)}.$$

Nonnegative shifted coefficients of the numerator certify the tail. Below the transport threshold, each negative finite seed imposes the cap $B \leq -n/\gamma_n - n^2 - An$. Taking

$$A = K_{18}^{(0)}, \quad B = \frac{1018970066171739632}{27274403735474723} = 37.35993923\dots$$

passes both tests and proves

$$(S1.10) \quad \Phi_n(H) \geq -\frac{n}{n^2 + K_{18}^{(0)}n + B} \quad (n \geq 8).$$

At $n = 8$ this bound equals γ_8 . It is slightly stronger than (S1.6); the latter keeps the coefficients small. For applications one can take the maximum of (S1.10), the applicable transported bounds, and the finite seeds.

Using the seed (S1.1) in the same construction gives

$$(S1.11) \quad \Phi_n(H) \geq -\frac{n}{n^2 + K_{18}n + B_*} \quad (n \geq 8), \quad B_* := \frac{800000000}{5274023} - 64 - 8K_{18} = 36.92163571\dots$$

The finite comparisons hold for $8 \leq n \leq 17$, with equality at 8. For the tail, the numerator in (S1.9) has positive constant and linear coefficients because $B_* > 0$ and $K_{18}^2 + 2K_{18} - B_* > 0$; its quadratic coefficient vanishes. These comparisons are verified exactly in the supporting files. Relative to (S1.10), the denominator increases by $(K_{18} - K_{18}^{(0)})(n - 8)$, so this bound is strictly stronger for every $n > 8$.

The same finite seeds give a bound with one formula in every dimension:

Corollary S1.5. *For $H \in \mathfrak{B}_n$ and $n \geq 2$,*

$$\Phi_n(H) \geq -\frac{n - 2}{n^2 + 4n + 8} \geq -\frac{1}{17}.$$

Proof. Dimensions two and three are nonnegative. For $4 \leq n \leq 7$, compare the exact seeds in Table 1 with the displayed fraction. For $n \geq 8$, Theorem S1.4 is stronger, since

$$-\frac{5n}{5n^2 + 31n + 190} + \frac{n - 2}{n^2 + 4n + 8} = \frac{n^2 + 88n - 380}{(5n^2 + 31n + 190)(n^2 + 4n + 8)} > 0.$$

Finally, $n^2 + 4n + 8 - 17(n - 2) = (n - 6)(n - 7) \geq 0$ for every integer $n \geq 2$. $\square$

The constant $-1/17$ is a convenient dimension-independent floor; no map attaining it is asserted.

S1.7. A conditional size-four seed.

Remark S1.6 (A stronger small-dimensional seed). *Suppose, conditionally, that $\Phi_4(H) \geq \gamma_4$ for every $H \in \mathfrak{B}_4$, with $\gamma_4 \in [-1/4, 0]$. Theorem 3.1 then gives $\Phi_n \geq h_n(K(4, \gamma_4))$ for $n \geq 7$, where*

$$K(4, \gamma_4) = \frac{3(1 + 4\gamma_4)}{1 - \gamma_4}.$$

In particular, a size-four nonnegativity theorem would give

$$\Phi_n(H) \geq -\frac{n-4}{n^2-n+3} \quad (n \geq 7).$$

Its transport constant 3 improves on the size-three zero seed's constant 2. The presently certified seed $\gamma_4 = -3/62$ gives $K = 30/13$. Both constants are smaller than K_{18} , so direct transport of either size-four seed does not improve the established rank-eighteen tail. A stronger size-four seed can also be imposed in every later finite program; this strengthens its constraints, but any improved numerical floor must be certified again.

S2. EXPLICIT RECURSIVE LOGARITHMIC BOUNDS

The paper proves the logarithmic coefficient $3e^2 - 7$ through the parity test (Proposition 4.6 of the paper). The results below are the three earlier explicit recurrences, with coefficients $12K/(k-1)$ at order n^{-6} , $4e - 10$, and $80(2e - 5)/(23 - 4e)$, each started from the dimension-eighteen seed. Their residual and projection lemmas are Lemmas B.1–B.3 of the paper.

S2.1. A quantitative projection obstruction.

Lemma S2.1 (A quartic projection inequality). *Let $H \in \mathfrak{B}_n$, let $1 \leq a < n$, and, for a partition λ padded with zeros to length n , put*

$$g_{a,\lambda} = \binom{n}{a}^{-1} \sum_{|I|=a} \left[4 \left(\sum_{i \in I} \lambda_i - a \right)^4 - 10 \left(\sum_{i \in I} \lambda_i - a \right)^2 + 3 \right].$$

Then its orbital weights satisfy

$$\sum_{\lambda \vdash n} g_{a,\lambda} w_\lambda \geq 0.$$

For the two leading shapes,

$$g_{a,(1^n)} = 3, \quad g_{a,(2,1^{n-2})} = 3 - \frac{12a(n-a)}{n(n-1)}.$$

Proof. For a rank- a projection P , the eigenvalues p_i of $H(P)$ lie in $[0, 1]$ and sum to the integer a . Let S be the sum of independent Bernoulli variables with parameters p_i . Put $v = \text{Var}(S)$ and $\mu_4 = \mathbb{E}(S - a)^4$. Independence gives

$$\mu_4 = 3v^2 + v - 6 \sum_i [p_i(1-p_i)]^2 \geq 3v^2 - \frac{1}{2}v.$$

Consequently,

$$4\mu_4 - 10v + 3 \geq 12v^2 - 12v + 3 = 3(2v-1)^2 \geq 0.$$

The probability generating polynomial of S is $\det(I + (z-1)H(P))$. Haar averaging over P identifies it with $f_H(z1_{[a]} + 1_{[n] \setminus [a]})$. On q_λ this substitution averages $z^{\sum_{i \in I} \lambda_i}$ over all a -element sets I . Applying the centered fourth- and second-moment functional therefore proves the row inequality. For (1^n) the centered exponent is zero. For $(2, 1^{n-2})$ it is 1 or -1 with total probability $2a(n-a)/(n(n-1))$, and zero otherwise, proving the last two formulas. $\square$

Proposition S2.2 (An explicit improvement over dimension transport). *Let $k \geq 3$, and let $\gamma \in (-1/k, 0]$ be a valid universal seed in dimension k . Put $K = K(k, \gamma) > 0$ as in (3.2). For $n \geq k + 3$, define*

$$(S2.1) \quad \begin{aligned} N &= n(n-1), & a &= \lfloor n/2 \rfloor, & m &= n-a, \\ \rho_{n,k,K} &= \frac{K}{(k-1)(N+K)} \left(n-k-2 - \frac{2(k-1)}{N} \right), \\ \Delta_{n,K} &= \frac{a(n-a)}{N+K} - \frac{1}{4}, \\ \varepsilon_{n,k,K} &= \frac{12(n-1)\rho_{n,k,K}\Delta_{n,K}}{n(4m^4 - 10m^2 + 8)}. \end{aligned}$$

Then $\rho_{n,k,K}, \Delta_{n,K}, \varepsilon_{n,k,K}$ are positive, and every $H \in \mathfrak{B}_n$ satisfies

$$\Phi_n(H) \geq h_n(K) + \varepsilon_{n,k,K}.$$

For fixed k, K , $\varepsilon_{n,k,K} \sim 12K n^{-6}/(k-1)$.

Proof. Write $\delta = \Phi_n(H) - h_n(K) \geq 0$. In the residual certificate (A.5), every residual outside (1^n) and $(2, 1^{n-2})$ is at least $\rho = \rho_{n,k,K}$, except that the residual at (n) vanishes when n is even. To verify this uniform lower bound, first set $\gamma = 0$. The endpoint argument for transport gives, for every such nonexceptional shape,

$$c_\lambda \geq \frac{n-k}{N+k-1} - \frac{2}{N}.$$

For selected even deficits the stronger bound $(n-k)/(N+k-1)$ is available; for selected odd deficits the subtraction of the mixed row only increases the residual. For unselected shapes the displayed bound follows from their scores being at least $-2/N$. The residual at $(n-1, 1)$ after the even-dimensional correction is $(n+1)(n-k+1)/(n(N+k-1))$, which is also larger than the displayed bound. At $\gamma = -1/k$ all residuals are nonnegative. The product Dc_λ is affine in γ , where $D = (1-\gamma)(N+K)$. Interpolation therefore gives

$$Dc_\lambda \geq (1+k\gamma) \left(n-k-2 - \frac{2(k-1)}{N} \right).$$

Since $(1+k\gamma)/(1-\gamma) = K/(k-1)$, this is $c_\lambda \geq \rho$. The bracket defining ρ is positive for $n \geq k+3$.

Let E be the total weight outside the two leading shapes. For odd n , the residual certificate gives $E \leq \delta/\rho$. For even n , its row $R_n \geq 0$ gives $w_{(n)} \leq w_{(n-1,1)}/(n-1)$, and hence in either parity

$$E \leq \frac{n}{n-1} \frac{\delta}{\rho}.$$

Put $y = w_{(2,1^{n-2})}$, $y_* = N/(N+K)$, and $v_a = a(n-a)/N$. Every score lies in $[-1/n, 1]$, so the score identity gives

$$y_* - y = \frac{n}{n+1} \left(\delta + \sum_{\lambda \neq (1^n), (2, 1^{n-2})} (1-r_\lambda) w_\lambda \right) \leq \frac{n}{n+1} \delta + E.$$

For every subset exponent, the centered value $T = \sum_{i \in I} \lambda_i - a$ lies in $[-a, n-a]$. Thus $|T| \leq m$, and, since $m \geq 3$,

$$g_{a,\lambda} - 3 \leq Q_n := 4m^4 - 10m^2.$$

The quartic row consequently implies

$$12v_a y \leq 3 + Q_n E.$$

Combining the last two estimates yields

$$12\Delta_{n,K} \leq \frac{12nv_a}{n+1}\delta + (Q_n + 12v_a)E \leq \left[\frac{12nv_a}{n+1} + \frac{n(Q_n + 12v_a)}{(n-1)\rho} \right] \delta.$$

Since $K/(k-1) \leq 1$ and the bracket defining ρ is smaller than n , we have $0 < \rho < 1/(n-1) < 1$. Also, $n/(n-1) \geq 1$, and $24v_a \leq 6n/(n-1) \leq 36/5 < 8$ for $n \geq 6$. The bracket is therefore at most $n(Q_n + 8)/((n-1)\rho)$, proving the claimed lower bound. Finally, $4a(n-a) - N$ equals n for even n and $n-1$ for odd n ; both exceed $K \leq k-1$. This proves positivity of $\Delta_{n,K}$. The stated asymptotic follows directly from the displayed formulas. $\square$

S2.2. Bounded projection tests and logarithmic improvement. The quartic test assigns coefficients of order n^4 to some shapes. Bounding those coefficients uniformly makes the gain large enough for logarithmic growth. The constant in Lemma S2.3 is best possible within its three-valued separator family. Theorem 4.8 bounds what the full recursive programs can certify from finitely many nonpositive initial bounds.

Lemma S2.3 (Optimal bounded projection separator). *Let S be a sum of independent Bernoulli variables whose mean a is an integer. Put*

$$C_{\text{pb}} = (2e - 5)^{-1}, \quad \psi_C(t) = \begin{cases} 1, & t = 0, \\ -1, & |t| = 1, \\ C, & |t| \geq 2. \end{cases}$$

Then $\mathbb{E}\psi_C(S - a) \geq 0$ for every such sum if and only if $C \geq C_{\text{pb}}$. The necessity concerns a constant uniform in the number of Bernoulli variables.

Proof. The hull argument for (2.8) applies to any linear functional of the coefficients of the probability generating polynomial. It therefore suffices to consider a deterministic integer shift of $\text{Bin}(r + s, r/(r+s))$, where r, s are positive integers, together with the wholly deterministic case. The latter gives value one. Write

$$q_{r,s} = \binom{r+s}{r} \left(\frac{r}{r+s} \right)^r \left(\frac{s}{r+s} \right)^s, \quad d_{r,s} = \frac{1}{r+1} + \frac{1}{s+1}.$$

The mass at the two neighbors of the mean is $(2 - d_{r,s})q_{r,s}$. Thus the desired inequality is equivalent to

$$F_C(r, s) := q_{r,s}[3C + 1 - (C + 1)d_{r,s}] \leq C.$$

It suffices to prove this for $C = C_{\text{pb}}$, since increasing C only increases the expected separator. The elementary bounds $5/2 < e < 11/4$ imply $C_{\text{pb}} > 2$.

First fix an integer $r \geq 2$ and extend s to real values at least two, using $\binom{s+r}{r} = \prod_{j=1}^r (s+j)/r!$. Set

$$A = 3C + 1 - \frac{C+1}{r+1}, \quad t = 1 - \frac{C+1}{A}.$$

For $C \geq 2$ one has $t \geq 1/2$, because $A \geq 2(C+1)$ is equivalent to $r(C-1) \geq 2$. Differentiation gives

$$\frac{\partial}{\partial s} \log F_C(r, s) = \sum_{j=2}^r \frac{1}{s+j} + \frac{1}{s+t} - \log \left(1 + \frac{r}{s} \right) \leq 0.$$

Indeed, the integral of $1/(s+x)$ on $[0, 1]$ is at least its midpoint value $1/(s+1/2)$ by convexity, and its integral on each $[j-1, j]$ is at least $1/(s+j)$. By symmetry, for integers $r, s \geq 2$,

$$F_C(r, s) \leq F_C(2, 2) = \frac{7C+1}{8} \leq C.$$

It remains to take $r = 1$. For real $s \geq 1$,

$$F_C(1, s) = \left(\frac{s}{s+1} \right)^s \frac{(5C+1)s + 3C - 1}{2(s+1)}.$$

Its logarithmic derivative is

$$-\log(1 + 1/s) + \frac{1}{s+t}, \quad t = \frac{3C-1}{5C+1}.$$

This derivative has the sign of $L(s) - t$, where

$$L(s) = \frac{1}{\log(1 + 1/s)} - s, \quad L'(s) = \frac{1}{s(s+1)\log^2(1 + 1/s)} - 1 > 0.$$

For the last inequality, write $x = \log(1 + 1/s) > 0$ and use $x < 2 \sinh(x/2) = 1/\sqrt{s(s+1)}$. Consequently $F_C(1, s)$ has no interior maximum; its supremum is at one of the two endpoints. Their values are

$$F_C(1, 1) = C, \quad \lim_{s \rightarrow \infty} F_C(1, s) = \frac{5C+1}{2e}.$$

The second is at most C exactly when $C \geq C_{\text{pb}}$. This proves sufficiency and, by the same binomial limit, necessity. $\square$

The rational choice $C = 23/10$ is valid because $e > \sum_{j=0}^6 1/j! = 1957/720 > 125/46$. The residual estimate in Lemma B.1 retains the positive contribution at the endpoint $\gamma = -1/k$.

Proposition S2.4 (A bounded projection correction). *Let $k \geq 3$ and let $\gamma \in (-1/k, 0]$ be a valid universal seed. Put $K = K(k, \gamma)$, take $n \geq k + 3$, and write*

$$N = n(n-1), \quad a = \lfloor n/2 \rfloor, \quad v = \frac{a(n-a)}{N}, \quad \Delta = \frac{a(n-a)}{N+K} - \frac{1}{4}.$$

For every $C \geq C_{\text{pb}}$, every $H \in \mathfrak{B}_n$ satisfies

$$(S2.2) \quad \Phi_n(H) \geq h_n(K) + \frac{4\Delta}{\frac{4nv}{n+1} + \frac{n(C-1+4v)}{(n-1)\tilde{\rho}_{n,k,K}}}.$$

For fixed k, K, C , the correction is asymptotic to $1/(Cn^2)$.

Proof. For the Bernoulli sum associated with $H(P)$, the generating-polynomial argument in Lemma S2.1 converts Lemma S2.3 into the orbital row

$$\sum_{\lambda \vdash n} g_{a,\lambda}^C w_\lambda \geq 0, \quad g_{a,\lambda}^C = \binom{n}{a}^{-1} \sum_{|I|=a} \psi_C \left(\sum_{i \in I} \lambda_i - a \right).$$

Its coefficients lie in $[-1, C]$, and the two leading coefficients are 1 and $1-4v$. Put $\delta = \Phi_n(H) - h_n(K)$, $y = w_{(2, 1^{n-2})}$, $y_* = N/(N+K)$, and let E be the total weight outside the two leading shapes. The residual and score arguments in Proposition S2.2, with the stronger floor from Lemma B.1, give

$$E \leq \frac{n\delta}{(n-1)\tilde{\rho}_{n,k,K}}, \quad y_* - y \leq \frac{n\delta}{n+1} + E.$$

Normalization and the bounded row imply $4vy \leq 1 + (C-1)E$. Since $4vy_* - 1 = 4\Delta$, combining these inequalities proves (S2.2). Positivity of Δ was checked in Proposition S2.2. For fixed parameters, $\tilde{\rho}_{n,k,K} \sim 1/n$, $\Delta \sim 1/(4n)$, and $v \rightarrow 1/4$, proving the asymptotic claim. $\square$

Proposition S2.5 (An explicit recursive logarithmic bound). *Suppose the bounded projection separator is valid with $\psi(0) = 1$, $\psi(\pm 1) = -1$, and $\psi(t) = C$ for $|t| \geq 2$, where $C \geq 2$. Start from*

$$k_0 = 18, \quad K_0 = \frac{6562108347902}{1034110757033}.$$

For $j \geq 0$, put $k_j = 18 + 3j$ and $n = k_j + 3$. Define

$$(S2.3) \quad \begin{aligned} a &= \lfloor n/2 \rfloor, \quad N = n(n-1), \quad v = \frac{a(n-a)}{N}, \quad \Delta = \frac{a(n-a)}{N+K_j} - \frac{1}{4}, \\ \varepsilon_j &= \frac{4\Delta}{\frac{4nv}{n+1} + \frac{n(C-1+4v)}{(n-1)\tilde{\rho}_{n,k_j,K_j}}}, \\ \kappa_j &= K(n, h_n(K_j) + \varepsilon_j), \quad K_{j+1} = \frac{\lfloor n^2 \kappa_j \rfloor}{n^2}. \end{aligned}$$

Then $h_{k_j}(K_j)$ is a valid universal seed in dimension k_j . For all $n \geq 21$, choosing $j = \lfloor (n-21)/3 \rfloor$ gives

$$\Phi_n(H) \geq h_n(K_j) \quad (H \in \mathfrak{B}_n).$$

Moreover,

$$K_j = \frac{2}{C} \log k_j + O(1).$$

Consequently, for some constant A independent of n, H ,

$$\Phi_n(H) \geq -\frac{1}{n} + \frac{(2/C) \log n - A}{n^2} \quad (n \geq 21).$$

Lemma S2.3 permits $C = C_{\text{pb}} = (2e-5)^{-1}$, giving coefficient $4e-10$. The rational choice $C = 23/10$ gives coefficient $20/23$. For rational C , every rounded seed in the recurrence is rational.

Proof. Proposition S2.4 gives the displayed correction whenever the current seed lies in $(-1/k, 0]$.

We check that the recurrence remains in the seed interval. For $n = k+3$ and $0 \leq K \leq k-1 = n-4$,

$$\frac{(n-3)(n-k)(n+k-2)}{n(n-2)} = \frac{3(n-3)(2n-5)}{n(n-2)} \leq 6.$$

The other numerator factor in $\tilde{\rho}$ is at most one. Therefore $\tilde{\rho} \leq 6/N$. Also $4v \geq 1$, $4\Delta \leq 1/(n-1)$, and hence

$$0 < \varepsilon_j \leq \frac{\tilde{\rho}}{Cn} \leq \frac{3}{n^2(n-1)}.$$

Since $-h_n(K) \geq 3/n^2$, the corrected seed remains negative and larger than $-1/n$. Thus $0 < \kappa_j < n-1$. The inverse transport identity gives

$$\kappa_j - K_j = \frac{\varepsilon_j(N+K_j)^2}{n^2 - 1 - \varepsilon_j(N+K_j)} \leq \frac{4}{n-1}.$$

Here $N+K_j \leq n^2$, and the denominator is at least $3n^2/4$ for $n \geq 21$. Rounding decreases this value by less than $1/n^2$. In particular $K_j \geq K_0 - \sum_{n \geq 21} n^{-2} > K_0 - 1/20 > 0$, so every step is valid, and monotonicity of h_n preserves the lower bound after rounding. Induction proves validity. The upper increment bound also gives $K_j = O(\log k_j)$.

For $n = k_j + 3$, use this logarithmic bound in the explicit formulas:

$$\tilde{\rho}_{n,k_j,K_j} = \frac{6}{n^2} + O\left(\frac{\log n}{n^3}\right), \quad \Delta = \frac{1}{4n} + O\left(\frac{\log n}{n^2}\right), \quad v = \frac{1}{4} + O(n^{-1}).$$

It follows that

$$\varepsilon_j = \frac{6}{Cn^3} + O\left(\frac{\log n}{n^4}\right), \quad K_{j+1} - K_j = \frac{6}{Cn} + O\left(\frac{\log n}{n^2}\right).$$

The rounding error is absorbed in the latter remainder. Summing along $n = 21, 24, 27, \dots$ gives $K_j = (2/C) \log k_j + O(1)$, since the remainder is summable. Transport from the largest $k_j \leq n - 3$ proves the stated bound, and $h_n(K) + 1/n = K(n+1)/(n(N+K))$ gives its asymptotic form. $\square$

S2.3. Pairing projection coefficients with individual residuals. Pairing each projection coefficient with its transport residual improves the coefficient in Proposition S2.5. The resulting recurrence is determined by thirteen small patterns.

Proposition S2.6 (The logarithmic coefficient from individual residuals). *Suppose the bounded separator is valid with $2 \leq C \leq 7/3$, and put*

$$M = \frac{13C - 2}{80}.$$

There is an absolute constant A_C such that every $H \in \mathfrak{B}_n$, $n \geq 21$, satisfies

$$\Phi_n(H) \geq -\frac{1}{n} + \frac{M^{-1} \log n - A_C}{n^2}.$$

In particular, the optimal bounded separator $C_{\text{pb}} = (2e - 5)^{-1}$ gives logarithmic coefficient

$$\frac{80}{13C_{\text{pb}} - 2} = \frac{80(2e - 5)}{23 - 4e}.$$

An explicit recurrence starts from $k_0 = 18$ and $K_0 = K_{18}$. For $k_j = 18 + 3j$, set $n = k_j + 3$, compute the pointwise correction $\varepsilon_j^* = 4\Delta/(\alpha_0 + M_{n,k_j,K_j})$, and define

$$(S2.4) \quad \begin{aligned} \varepsilon_j &= \min\{\varepsilon_j^*, 30/n^3, -h_n(K_j)/2\}, & \kappa_j &= K(n, h_n(K_j) + \varepsilon_j), \\ K_{j+1} &= \frac{\lfloor n^2 \kappa_j \rfloor}{n^2}. \end{aligned}$$

Every rounded seed is valid; for rational C it is rational.

Proof. The cap ensures the corrected seed lies in $(-1/n, 0)$, and

$$0 \leq \kappa_j - K_j = \frac{\varepsilon_j(N + K_j)^2}{n^2 - 1 - \varepsilon_j(N + K_j)} \leq \frac{31}{n} \quad (n \geq 21).$$

Rounding loses less than $1/n^2$. Thus K_j stays positive, bounded below by $K_0 - 1/20$, and is $O(\log k_j)$. Validity follows inductively from Lemma B.3 and Theorem 3.1.

We estimate the pointwise maximum when $k = n - 3$ and $K = O(\log n)$. The endpoint floor in Lemma B.2 and affine interpolation with the nonnegative $\gamma = 0$ endpoint give, uniformly over all selected shapes with s coordinates different from one,

$$c_\lambda \geq \frac{3 \min(s-1, 7)}{n^2} - O\left(\frac{\log n}{n^3}\right).$$

Indeed the interpolation factor of the first endpoint is $(1 - K/(k-1))N/(N+K) = 1 - O(\log n/n)$, and $1 - (1 - 3/n)^{s-1} \geq 1 - (1 - 3/n)^{\min(s-1, 7)}$. For each fixed pattern of these coordinates, the exact residual formula also gives

$$c_\lambda = \frac{3(s-1)}{n^2} + O\left(\frac{\log n}{n^3}\right).$$

All unselected residuals, including the corrected parity partner, are $1/n + O(\log n/n^2)$ or larger. Their effective coefficients are bounded, so their ratios are $O(n)$.

For a fixed number s of these coordinates, balanced sampling without replacement tends with error $O(1/n)$ to independent fair sampling on those s coordinates. If g_μ denotes the resulting

expectation of ψ , then $\widehat{d}_\lambda = g_\mu + O(1/n)$ for each fixed nonleading shape. Writing μ for the parts exceeding one, the complete list for $3 \leq s \leq 7$ is

μ	g_μ	μ	g_μ
(3)	$(C-1)/4$	(2, 2)	$(C-1)/8$
(4)	$(2C-1)/4$	(2, 3)	$(5C-3)/16$
(5)	$(11C-3)/16$	(2, 2, 2)	$(7C-5)/32$
(2, 4)	$(8C-3)/16$	(3, 3)	$(13C-5)/32$
(6)	$(13C-2)/16$	(2, 2, 3)	$(23C-11)/64$
(2, 5)	$(21C-5)/32$	(3, 4)	$(33C-9)/64$
(7)	$(57C-5)/64$		

Every entry satisfies $g_\mu/(s-1) \leq M$ for $2 \leq C \leq 7/3$; these linear inequalities can be checked at the two interval endpoints. Equality holds at (6). For $s \geq 8$, the universal bound $b_\lambda \leq C$, the estimate $d_\lambda = b_\lambda - r_\lambda + O(n^{-2})$, and $r_\lambda \geq -2/N$ in the selected range give the ratio bound with constant $C/7 < M$. Consequently

$$M_{n,n-3,K} = \frac{n^2}{3} \left(M + O\left(\frac{\log n}{n}\right) \right).$$

Since $4\Delta = 1/n + O(\log n/n^2)$ and $\alpha_0 = 1 + O(n^{-2})$,

$$\varepsilon_j^* = \frac{3}{Mn^3} + O\left(\frac{\log n}{n^4}\right).$$

Here $3/M \leq 10$, so both caps are eventually inactive. The inverse transport formula and rounding now give

$$K_{j+1} - K_j = \frac{3}{Mn} + O\left(\frac{\log n}{n^2}\right).$$

The remainder is summable; summing over every third dimension yields $K_j = M^{-1} \log k_j + O(1)$. Transport to each intervening dimension and $h_n(K) + 1/n = K(n+1)/(n(N+K))$ prove the assertion. $\square$

S3. A THREE-SHAPE FEASIBLE PROFILE

Theorem 4.8 of the paper bounds the logarithmic coefficient certifiable by the recursive relaxation by $3e^2 - 7$, through the binary profiles of Lemma S4.3. The explicit three-shape profile below, with cutoff at dimension 64, gives the weaker bound 24 and was the first such obstruction.

S3.1. Feasible profiles with logarithmic correction. The three-shape construction fixes the cutoff at dimension 64, but gives the weaker upper bound 24 on the logarithmic coefficient certified by the recursive programs. Theorem 4.8 lowers that bound to $3e^2 - 7$.

Proposition S3.1 (A limitation of the full recursive relaxation). *Let $H_j = \sum_{i=1}^j i^{-1}$ and*

$$\Lambda_n = 63 + 24(H_{n-4} - H_{60}) \quad (n \geq 64).$$

Consider recursive coefficient programs using nonnegative normalized weights, the square-free floor, scalar capacity, all projection-spectrum hull constraints, and all proper-dimensional mixed rows. Starting from any nonpositive seeds below dimension 64, every lower bound γ_n certified recursively by these programs satisfies

$$\gamma_n \leq h_n(\Lambda_n) = -\frac{1}{n} + \frac{24 \log n + O(1)}{n^2} \quad (n \geq 64).$$

Thus these constraints cannot improve the leading constant 1 or prove a correction larger in order than $\log n/n^2$.

Lemma S3.2 (Feasible profiles for the full recursive relaxation). *Write $\mathbf{H}_j = \sum_{i=1}^j i^{-1}$ and define*

$$\Lambda_k = k - 1 \quad (2 \leq k \leq 64), \quad \Lambda_n = 63 + 24(\mathbf{H}_{n-4} - \mathbf{H}_{60}) \quad (n \geq 64).$$

For $n \geq 64$, put $N = n(n-1)$ and

$$z_n = \frac{8}{n}, \quad x_n = \frac{\Lambda_n}{N + \Lambda_n} - \frac{z_n}{n-1}, \quad y_n = \frac{N}{N + \Lambda_n} - \frac{(n-2)z_n}{n-1}.$$

The symmetric polynomial

$$\mathcal{F}_n = x_n q_{(1^n)} + y_n q_{(2, 1^{n-2})} + z_n q_{(2, 2, 1^{n-4})}$$

has nonnegative normalized coefficients, capacity one, and satisfies the square-free coefficient floor and every projection-spectrum hull constraint. Moreover, for every $2 \leq k < n$ and every $\gamma_k \leq h_k(\Lambda_k)$,

$$(\mathcal{D}_k - \gamma_k \mathcal{J}_k) \mathcal{F}_n$$

is coefficientwise nonnegative. Here $h_k(\Lambda_k) = 0$ for $k \leq 64$. Its score is

$$\mathcal{L}_n(\mathcal{F}_n) = h_n(\Lambda_n) = -\frac{1}{n} + \frac{24 \log n + O(1)}{n^2}.$$

Proof. The increments of Λ_n after dimension 64 are $24/(n-4) < 1$. Consequently $63 \leq \Lambda_n \leq n-1$ for $n \geq 64$. The weights sum to one, $x_n < 1/n$, $y_n \geq 1 - 9/n$, and

$$x_n \geq \frac{63}{n^2} - \frac{8}{n(n-1)} > \frac{54}{n^2} > 0.$$

In particular they are nonnegative. Symmetry and normalization imply capacity one by the arithmetic-geometric mean inequality applied to each monomial orbit. Also $n!/n^n \leq 1/n^2$ for $n \geq 5$: the quantity $n^2 n!/n^n$ is at most one at $n = 5$, and its successive ratio is $(1 + 1/n)^{2-n} < 1$. This proves the square-free floor. The displayed score follows directly from $r_{(1^n)} = 1$, $r_{(2, 1^{n-2})} = -1/n$, and $r_{(2, 2, 1^{n-4})} = 2/N$.

We next verify the full projection hull, not merely finitely many moments. Fix a rank $1 \leq a \leq n-1$, put $X = a(n-a)$, and center the exponent of $\mathcal{F}_n(z \mathbf{1}_a + \mathbf{1}_{n-a})$ at its mean a . The resulting distribution is symmetric and supported on $\{-2, -1, 0, 1, 2\}$. Its masses at each of ± 1 and ± 2 are

$$p_1 = v y_n + q z_n, \quad p_2 = r z_n,$$

where

$$v = \frac{X}{N}, \quad q = \frac{2X(n^2 - 3n + 4 - 2X)}{N(n-2)(n-3)}, \quad r = \frac{X(X-n+1)}{N(n-2)(n-3)}.$$

It is a mixture of the centered deterministic distribution, the centered $\text{Bin}(2, 1/2)$ distribution, and the two centered distributions $\text{Bin}(3, 1/3)$ and $\text{Bin}(3, 2/3)$, with weights

$$\pi_0 = 1 - 4p_1 + 2p_2, \quad \pi_2 = 4p_1 - 56p_2, \quad \pi_{3,-} = \pi_{3,+} = 27p_2.$$

These weights sum to one. Since $r/v \leq (n-2)/(4(n-3))$ and $q \geq 0$,

$$p_1 - 14p_2 \geq v \left(1 - \frac{9}{n} - \frac{28(n-2)}{n(n-3)} \right) \geq v(1 - 38/n) > 0.$$

Thus $\pi_2 \geq 0$. To check π_0 , regard $P = p_1 - p_2/2$ as a function of X . Its derivative has the sign of

$$y_n + \frac{z_n}{(n-2)(n-3)} \left(2n^2 - \frac{11}{2}n + \frac{15}{2} - 9X \right).$$

For $X \leq n^2/4$ and $n \geq 64$, the parenthesized expression is at least $-(n-2)(n-3)$. Hence P increases with X , because $y_n - z_n \geq 1 - 17/n > 0$. At a balanced rank, the condition $P \leq 1/4$ is respectively

$$x_n + \frac{n+6}{8(n-3)} z_n \geq \frac{1}{n} \quad (n \text{ even}), \quad x_n + \frac{n+7}{8(n-2)} z_n \geq \frac{1}{n+1} \quad (n \text{ odd}).$$

Both hold already with $x_n = 0$ and $z_n = 8/n$. Thus $\pi_0 \geq 0$. All four distributions are projection-hull generators: the two cubic ones are available when $2 \leq a \leq n-2$, and at the endpoint ranks $p_2 = 0$, so their mixture weights vanish. The ranks zero and n are wholly deterministic. This proves every projection-spectrum constraint.

For the mixed constraints, write

$$\mathcal{F}_n = ce_2e_{n-2} + be_1e_{n-1} + de_n,$$

where

$$c = \frac{4z_n}{N(n-2)(n-3)}, \quad b = \frac{y_n}{N} - \frac{4z_n}{N(n-3)}, \quad d = x_n - \frac{y_n}{n-1} + \frac{2z_n}{(n-1)(n-2)}.$$

We have $c \geq 0$ and $b \geq 0$, since $y_n - 4z_n/(n-3) \geq 1 - 10/n > 0$. Split the variables into active $x = (x_1, \dots, x_k)$ and inactive $y = (y_1, \dots, y_m)$, where $m = n - k$. By the Pieri rule, products of elementary symmetric polynomials are Schur-positive in the active variables, and $\mathcal{D}_k$ preserves Schur positivity:

$$\mathcal{D}_k s_\lambda = \left[\prod_{i=1}^k (\lambda_i + k - i) \right] s_{\lambda - (1^k)}$$

when $\lambda_k \geq 1$, and is zero otherwise. Ordinary differentiation preserves nonnegative monomial coefficients. Therefore, for $\gamma \leq 0$, every coefficient involving a positive active degree is nonnegative; the only possibly negative summand, de_n , contributes only active degree zero. In active degree zero, the terms other than $e_m(y)$ have nonnegative coefficients. Thus only the coefficient of the square-free inactive monomial needs checking.

After division by $k!$, this coefficient is

$$c[1 + m + \binom{m}{2} - \gamma(\binom{k}{2}^2 + mk^2 + \binom{m}{2})] + b[m + 1 - \gamma(k^2 + m)] + d(1 - \gamma).$$

Take $\gamma = h_k(\Lambda_k)$. Substituting the three weights and dividing by $1 - \gamma > 0$ simplifies this coefficient to

$$J = x_n - \frac{\Lambda_k}{N} y_n + \frac{z_n(k-2)(k-3)}{n(n-1)(n-2)(n-3)} \left(k - \frac{\Lambda_k(k-2)}{k-1} \right).$$

Set $A = (k-2)(k-3)/((n-2)(n-3))$. Then

$$NJ = \frac{N}{N + \Lambda_n} (\Lambda_n - \Lambda_k) - z_n B,$$

where

$$B = n - Ak - \Lambda_k \left(\frac{n-2}{n-1} - A \frac{k-2}{k-1} \right) \leq n - Ak.$$

The inequality uses $\Lambda_k \geq 0$ and the nonnegativity of the expression in parentheses; it holds also for $k = 2, 3$, when $A = 0$. It is therefore enough to prove

$$\Lambda_n - \Lambda_k \geq 8 \left(1 + \frac{\Lambda_n}{N} \right) \left(1 - \frac{k(k-2)(k-3)}{n(n-2)(n-3)} \right).$$

For $k \leq 32$, the left side is at least 32, whereas the right side is at most $8(1 + 1/n)$. For $k \geq 33$, each increment from k to n is at least $24/(j-4)$, including the unit increments before dimension 64. Hence

$$\Lambda_n - \Lambda_k \geq \frac{24(n-k)}{n-4}.$$

On the other hand, for $k \geq 3$, the elementary product inequality $1 - uvw \leq (1-u) + (1-v) + (1-w)$ for $u, v, w \in [0, 1]$ gives

$$1 - \frac{k(k-2)(k-3)}{n(n-2)(n-3)} \leq \frac{3(n-k)}{n-3}.$$

Since $1/(n-4) \geq (1+1/n)/(n-3)$, the sufficient inequality follows. This proves all mixed constraints at $\gamma = h_k(\Lambda_k)$. Smaller seeds preserve them because $\mathcal{T}_k \mathcal{F}_n$ has nonnegative coefficients.

Finally, begin with any nonpositive seeds in dimensions below 64. They are bounded above by $h_k(\Lambda_k) = 0$. Inductively the displayed feasible profile prevents the recursive relaxation in each dimension $n \geq 64$ from certifying a value exceeding $h_n(\Lambda_n)$, so every later seed satisfies the hypothesis just verified. Since $\Lambda_n = 24 \log n + O(1)$ and $h_n(\Lambda_n) + 1/n = \Lambda_n(n+1)/(n(N+\Lambda_n))$, Proposition S3.1 follows. $\square$

S4. LIMITS OF THE COEFFICIENT RELAXATION

S4.1. Binary profiles limiting the recursive coefficient. The mean-one binomial generators used for the separator of Proposition 4.6 also produce feasible profiles for the full recursive relaxation. Their positive expansion in balanced binary laws gives the coefficients of these profiles.

Lemma S4.1 (Positive binary expansion of the binomial generator). *For an integer $m \geq 3$, put $p = 1/m$, $a = 1 - 2p$, and*

$$F(z) = z^{-1}(1 - p + pz)^m, \quad G(w) = \frac{F(z) + F(z^{-1})}{2}, \quad w = \frac{z + 2 + z^{-1}}{4}.$$

Then G is a polynomial of degree $m - 1$,

$$G(w) = \sum_{r=0}^{m-1} a_r w^r,$$

with $a_0 = -a^m < 0$ and $a_r > 0$ for every $1 \leq r \leq m - 1$. In particular,

$$a_1 = \frac{2(m-1)(3m-4)}{m^2} a^{m-2}.$$

The coefficient positivity is proved in Section S4.2 by an alternating-sum formula whose successive absolute terms decrease. The following calculation identifies the cost of the positive coefficients in the mixed inequalities.

Corollary S4.2 (Limiting coefficient of binary profiles). *Let $R = m - 1$ and $C_R = \sum_{r=2}^R (2r - 1)a_r / (-a_0)$. Then*

$$C_R = \frac{3R-1}{R+1} \left(\frac{R+1}{R-1} \right)^{R+1} - 1 - \frac{2R(3R-1)}{(R-1)^2}, \quad \lim_{R \rightarrow \infty} C_R = 3e^2 - 7.$$

Proof. The centered binomial distribution has variance $R/(R+1)$. Consequently $G(1) = 1$ and $G'(1) = 2R/(R+1)$, so

$$\sum_{r=2}^R (2r-1)a_r = 2G'(1) - G(1) + a_0 - a_1 = \frac{3R-1}{R+1} + a_0 - a_1.$$

Insert the preceding formulas for a_0, a_1 and take the limit. $\square$

For each fixed R , normalize the positive coefficients by $t_r = a_r / (-a_0)$, $2 \leq r \leq R$. Lemma S4.3 lifts this expansion to profiles with weights t_r/n on $(2^r, 1^{n-2^r})$. The remaining mass lies on the two leading shapes, and their balance is chosen so that the score is $h_n(L_n)$, with

$$L_n = (C_R + \eta) \log n + O(1), \quad \eta > 0.$$

The positive expansion supplies the projection-hull decompositions; the harmonic increments of L_n dominate the loss in every proper-dimensional mixed row. Section S4.2 verifies these two requirements uniformly over the projection rank and the compression dimension.

Proof of Theorem 4.8. Lemma S4.3 constructs feasible profiles with logarithmic coefficient $C_R + \eta$ for every fixed $R \geq 2$ and $\eta > 0$. Corollary S4.2 gives $C_R \rightarrow 3e^2 - 7$. First let $n \rightarrow \infty$, then $\eta \downarrow 0$ and $R \rightarrow \infty$. $\square$

Proposition S4.4 records the separate obstruction when only mixed rows are retained.

S4.2. Positive binary coefficients and feasible profiles.

Proof of Lemma S4.1. Symmetry in z and z^{-1} makes G a polynomial in $z + z^{-1} = 4w - 2$. At $w = 0$ we have $z = -1$, giving $a_0 = -a^m$. For the other coefficients, put $t = z + 1$, so $w = t^2/(4(t-1))$. The coefficient-residue formula is

$$a_r = \frac{4^r}{2} [t^{2r}] (t-1)^{r-2} (t-2) (a+pt)^m.$$

To justify it, a small circle about $t = 0$ maps twice around $w = 0$. The two contributions $F(z)$ and $F(z^{-1})$ have equal contour integrals by the local involution $z \mapsto z^{-1}$, which preserves w . Thus

$$a_r = \frac{1}{2} \operatorname{Res}_{t=0} F(t-1) w(t)^{-r-1} w'(t),$$

and substitution gives the coefficient formula. For $r = 1$, use $(t-2)/(t-1) = 2 + t + t^2 + \dots$ in this formula to obtain the displayed positive formula for a_1 .

For $r \geq 2$, expand

$$(t-1)^{r-2} (t-2) = \sum_{l=0}^{r-1} (-1)^l A_l t^{r-1-l}, \quad A_l = \frac{r-1+l}{r-1} \binom{r-1}{l}.$$

It follows that

$$a_r = \frac{4^r}{2} \sum_{l=0}^{\min(r-1, m-r-1)} (-1)^l T_l, \quad T_l = A_l \binom{m}{r+1+l} a^{m-r-1-l} p^{r+1+l} > 0.$$

The successive terms decrease strictly, since

$$\frac{T_{l+1}}{T_l} = \frac{r+l}{r-1+l} \frac{r-1-l}{l+1} \frac{m-r-1-l}{r+2+l} \frac{1}{m-2} < \frac{r}{r+2} < 1.$$

Indeed the first factor is at most $r/(r-1)$, the second at most $r-1$, the denominator $r+2+l$ is at least $r+2$, and $(m-r-1-l)/(m-2) < 1$. Pairing adjacent positive and negative terms proves that the finite alternating sum is positive. $\square$

Lemma S4.3 (Feasible profiles from a binomial expansion). *Fix $R \geq 2$. Let G be a symmetric projection-hull generator supported on $\{-R, \dots, R\}$, available whenever both the rank and corank are at least R . Write Q_r for the centered $\operatorname{Bin}(2r, 1/2)$ law, with Q_0 the point mass at zero. Its generating polynomial is $Q_r(u) = u^r$, where $u = (z + 2 + z^{-1})/4$. Suppose that*

$$AG(u) = -1 + bu + \sum_{r=2}^R t_r u^r, \quad A, b, t_2, \dots, t_R > 0.$$

Put $T = \sum_{r=2}^R t_r$ and $C = \sum_{r=2}^R (2r-1)t_r$. For every $\varepsilon > 0$, there is an integer n_0 such that the following profiles, for every $n \geq n_0$, are feasible for all the recursive mixed rows, all projection spectrum hulls, capacity one, and the square-free coefficient floor. Set

$$L_k = k - 1 \quad (2 \leq k \leq n_0), \quad L_n = n_0 - 1 + (C + \varepsilon)(H_n - H_{n_0}) \quad (n \geq n_0),$$

write $N = n(n-1)$, and define

$$z_{r,n} = \frac{t_r}{n}, \quad \eta_r(n) = \frac{1 + n(-1)^r / \binom{n}{r}}{n+1},$$

$$x_n = \frac{L_n}{N + L_n} - \sum_{r=2}^R \eta_r(n) z_{r,n}, \quad y_n = 1 - x_n - \frac{T}{n},$$

$$F_n = x_n q(1^n) + y_n q(2, 1^{n-2}) + \sum_{r=2}^R z_{r,n} q(2^r, 1^{n-2r}).$$

The mixed rows hold for every proper-dimensional seed $\gamma_k \leq h_k(L_k)$. The score is

$$\mathcal{L}_n(F_n) = h_n(L_n) = -\frac{1}{n} + \frac{(C + \varepsilon) \log n + O(1)}{n^2}.$$

Proof. All constants below may depend on the fixed coefficients and on ε , but not on n , the projection rank, or the mixed-row dimension. We choose n_0 after these constants have been fixed. For sufficiently large n , $0 < \eta_r(n) \leq 2/n$. Choose $n_0 > C + \varepsilon + 1$ so that $n_0 - 1 \leq L_n \leq n - 1$. Then

$$0 < x_n < 1/n, \quad y_n \geq 1 - (1 + T)/n, \quad x_n \geq \frac{n_0 - 1 - 2T}{n^2}.$$

Increasing n_0 makes all weights nonnegative and gives $x_n \geq 1/n^2 \geq n!/n^n$. Symmetry and normalization give capacity one. The score formula follows from the definition of $\eta_r(n)$, since $\eta_r(n) = [r_{(2^r, 1^{n-2r})} + 1/n]/(1 + 1/n)$.

We verify the entire projection hull uniformly over all ranks. Fix a , set $b' = n - a$, $\theta = 4ab'/n^2$, and $\theta_n = 4ab'/N = n\theta/(n - 1)$. For the binary orbit with r doubled coordinates, its exact centered probability generating polynomial is

$$(L1) \quad P_{r,n,a}(u) = \sum_{\ell=0}^r \binom{r}{\ell} 4^\ell v_\ell (u - 1)^\ell, \quad v_\ell = \frac{(a)_\ell (b')_\ell}{(n)_{2\ell}}.$$

This identity follows by expanding the generating function for uniformly sampling the r positive and r negative marked coordinates. Its coefficient of u^j is

$$(L2) \quad d_{r,j} = \binom{r}{j} 4^j v_j S_{r,j}, \quad S_{r,j} = \sum_{h=0}^{r-j} \binom{r-j}{h} (-4)^h \frac{(a-j)_h (b'-j)_h}{(n-2j)_{2h}}.$$

When $j \leq \min(a, b')$, $S_{r,j}$ is the expected product of $2(r-j)$ signs sampled without replacement from a population of $n - 2j$ signs, of which $a - j$ are negative. Coupling with independent sampling gives, uniformly in the rank,

$$(L3) \quad S_{r,j} = \left(1 - \frac{2(a-j)}{n-2j}\right)^{2(r-j)} + O_R(1/n), \quad S_{r,j} \geq -O_R(1/n).$$

Indeed the coupling fails with probability at most $\binom{2(r-j)}{2}/(n-2j)$, and the product of signs has absolute value one. Also $4^j v_j = \theta^j + O_R(1/n)$ uniformly, by the same finite-sampling comparison. If $j > \min(a, b')$, every $d_{r,j}$ is zero.

Write the generating polynomial of F_n as $\sum_{j=0}^R c_j u^j$. For $j \geq 2$, (L2)–(L3) and $S_{j,j} = 1$ imply

$$(L4) \quad c_j \geq 4^j v_j \left(\frac{t_j}{n} - \frac{D_1}{n^2}\right) \geq 0, \quad c_j \geq \frac{t_j \theta^R}{n} - \frac{D_2}{n^2},$$

for fixed constants D_1, D_2 and all sufficiently large n . Choose D large enough that

$$\tau = \max \left\{ 0, \frac{\theta^R}{n} - \frac{D}{n^2} \right\}$$

satisfies $c_j \geq t_j \tau$ for every $j \geq 2$. We may also ensure $\tau = 0$ whenever $\min(a, b') < R$: in that case $\theta \leq 4R/n$, so $\theta^R/n \leq (4R)^R/n^{R+1}$. Use the assumed expansion to replace $\tau \sum_{j=2}^R t_j Q_j$ by $A\tau G + \tau Q_0 - b\tau Q_1$. The resulting hull weights are

$$c_0 + \tau, \quad c_1 - b\tau, \quad c_j - t_j \tau \quad (2 \leq j \leq R), \quad A\tau.$$

They sum to one. We check the first two weights next.

Since the $r = 1$ term is exactly $1 - \theta_n + \theta_n u$, (L2)–(L3) give

$$c_1 \geq \theta_n \left(y_n - \frac{D_3}{n^2} \right).$$

As $\tau \leq \theta^R/n \leq \theta_n/n$, it follows that $c_1 - b\tau \geq 0$ for all sufficiently large n . For the constant coefficient, (L3) with $j = 0$ gives

$$c_0 \geq x_n + y_n(1 - \theta_n) - D_4/n^2.$$

Since $y_n = 1 - x_n - T/n$ and $\tau \geq \theta^R/n - D/n^2$, we obtain

$$(L5) \quad c_0 + \tau \geq (1 - \theta) \left(1 - \frac{D_5}{n} \right) + x_n \theta_n - \frac{D_6}{n^2}.$$

Here one uses $\theta - \theta^R \leq (R - 1)(1 - \theta)$ and $\theta_n - \theta = \theta/(n - 1)$. When $\theta \leq 1/2$, the first term in (L5) dominates the error for large n . When $\theta \geq 1/2$, the second is at least $(n_0 - 1 - 2T)/(2n^2)$; choose $n_0 > 2D_6 + 2T + 1$. Thus all hull weights are nonnegative. The generator Q_j is available precisely when $j \leq \min(a, b')$, and unavailable weights vanish by (L2). The generator G is used only when $\min(a, b') \geq R$, by our choice of τ . Ranks zero and n are deterministic, which proves the hull claim.

For the mixed rows, use the triangular identities

$$e_j e_{n-j} = \sum_{r=0}^j \binom{n-2r}{j-r} m_{(2r, 1^{n-2r})}$$

to expand $F_n = \sum_{j=1}^R \kappa_j e_j e_{n-j} + de_n$. Backward substitution shows that every κ_j , $j \geq 2$, has positive leading term

$$\kappa_j = (j!)^2 t_j n^{-2j-1} (1 + O(1/n)),$$

because contributions from indices greater than j are smaller by at least one power of n . Also $\kappa_1 = n^{-2}(1 + O(1/n)) > 0$. Thus all these coefficients are positive for large n . The Pieri rule gives Schur positivity of the products in the active variables, and (A.4) shows that $\mathcal{D}_k$ preserves it. Together with coefficient positivity of ordinary differentiation, this shows that every mixed coefficient except the square-free inactive one is nonnegative for $\gamma_k \leq 0$.

For the remaining coefficient, define at integers $k \geq 2r$

$$A_r(k) = \frac{(k)_{2r}}{k+1} \left(1 + \frac{k(-1)^r}{\binom{k}{r}} \right), \quad B_r(k) = \frac{(k)_{2r}}{k^2-1} \left(1 - \frac{(-1)^r}{\binom{k}{r}} \right),$$

and set both values to zero for $k < 2r$. For large real t , $A_r(t)$ is a monic polynomial of degree $2r - 1$, while $B_r(t)$ is rational with $B_r(t) = t^{2r-2}(1 + O(1/t))$ and $B'_r(t) = (2r - 2)t^{2r-3}(1 + O(1/t))$. Their numerators before division are respectively

$$(t)_{2r} + (-1)^r r! t(t - r)_r, \quad (t)_{2r} - (-1)^r r! (t - r)_r.$$

The first numerator vanishes at $t = -1$, which proves the polynomial claim for A_r . The values $A_r(k), B_r(k)$ are nonnegative at every integer $k \geq 2r$. For sufficiently large n , eventual monotonicity of the fixed rational function B_r gives $B_r(k) \leq B_r(n)$ for every $k < n$, including the finitely many smaller k .

At $\gamma_k = h_k(L_k)$, the square-free inactive coefficient divided by $k!(1 - \gamma_k)$ is

$$J = x_n - \frac{L_k}{N} y_n + \sum_{r=2}^R \frac{z_{r,n}}{(n)_{2r}} [A_r(k) - L_k B_r(k)].$$

Since

$$\eta_r(n) = \frac{A_r(n)}{(n)_{2r}}, \quad 1 - \eta_r(n) = \frac{NB_r(n)}{(n)_{2r}},$$

substitution and the preceding monotonicity give

$$(L6) \quad NJ \geq \frac{N}{N + L_n}(L_n - L_k) - \sum_{r=2}^R \frac{t_r}{n} N \eta_r(n) \left(1 - \frac{A_r(k)}{A_r(n)}\right).$$

Fix a large integer K . For $k \geq K$, polynomial derivative bounds, uniformly for $k < n$, give

$$(L7) \quad 1 - \frac{A_r(k)}{A_r(n)} \leq (2r - 1 + O(1/n)) \frac{n - k}{n}, \quad \frac{N \eta_r(n)}{n} = 1 + O(1/n).$$

Indeed, if $d = 2r - 1$, monicity gives $A'_r(t) \leq dn^{d-1} + O(n^{d-2})$ uniformly for $K \leq t \leq n$. Integrating and dividing by $A_r(n) = n^d(1 + O(1/n))$ proves (L7). Choose n_0 large enough that the subtracted expression in (L6), after multiplication by $1 + L_n/N$, is at most $(C + \varepsilon)(n - k)/n$. We may choose $K > C + \varepsilon$. Every increment of L from such k to n is at least $(C + \varepsilon)/j$, including unit increments before n_0 , and hence

$$L_n - L_k \geq (C + \varepsilon) \frac{n - k}{n}.$$

For the remaining finitely many $k < K$, the subtracted expression in (L6) is bounded independently of n_0 , whereas $L_n - L_k \geq n_0 - K$. Increasing n_0 therefore proves $J \geq 0$ for all proper k . Smaller seeds preserve the rows because $\mathcal{T}_k F_n$ has nonnegative coefficients. To start the induction at an arbitrarily large cutoff, use the zero-valued bistochastic map in dimension three from the proof of Proposition S6.6. By (5.1), its pinched identity extensions have score zero in every dimension $n \geq 3$. Actual maps are feasible for each program, so every recursively certified seed remains nonpositive until the chosen cutoff; the dimension-two seed is nonpositive by hypothesis. Induction then gives the stated recursive obstruction. Finally $L_n = (C + \varepsilon) \log n + O(1)$ gives its asymptotic score. $\square$

S4.3. An obstruction using only mixed coefficients.

Proposition S4.4 (An obstruction for all proper mixed rows). *For every $n \geq 4$, the polynomial*

$$F_n(z) = \frac{e_1(z)e_{n-1}(z) - 2e_n(z)}{n^2 - 2} = f_{n,n-2}(z)$$

has nonnegative coefficients, value one at 1^n , scalar capacity one, and square-free coefficient at least $n!/n^n$. It satisfies

$$(\mathcal{D}_k - \gamma_k \mathcal{T}_k) F_n \geq_{\text{coeff}} 0 \quad (2 \leq k < n, \gamma_k \leq 0), \quad \mathcal{L}_n(F_n) = -\frac{1}{n^2 - 2}.$$

Yet F_n is not the orbital polynomial of any map in $\mathfrak{B}_n$.

Proof. The capacity and coefficient claims follow from the proof of Theorem 4.1 with $K = n - 2 \geq 2$. In (B.1) at $\gamma = 0$, the first coefficient is $K - (k - 1) = n - k - 1 \geq 0$, so $\mathcal{D}_k F_n \geq_{\text{coeff}} 0$. Since F_n has nonnegative coefficients, so does $\mathcal{T}_k F_n$. This proves the assertion for every nonpositive seed. The two weights give $h_n(n - 2) = -1/(n^2 - 2)$, and Proposition 4.2 excludes realizability. $\square$

This is an obstruction for the relaxation: even with scalar capacity and the square-free floor, the mixed rows cannot certify an $o(n^{-2})$ lower bound on feasible polynomials. Projection compatibility excludes the displayed profile from actual maps.

S4.4. A limitation of bounds for individual stable tuples. The orbital representation of Φ_n averages signed scores over all input bases. An estimate applied separately to each balanced positive tuple therefore faces a different extremal problem. The example below gives such a tuple with score of order $-1/n$, even though the associated positive map has a positive invariant. We call the tuple balanced when $A_i \succeq 0$, $\text{tr } A_i = 1$, and $\sum_i A_i = I$. A real polynomial is stable if it has no zero when all variables have positive imaginary part.

For $n \geq 2$, use the coefficient functional $\mathcal{L}_n$ from (2.2). Put $m = n - 1$ and

$$W = \{z \in \mathbb{C}^m : \sum_j z_j = 0\}, \quad v_j = e_j - m^{-1}\mathbf{1} \in W.$$

On $\mathbb{C}^2 \oplus W$, define

$$A_1 = \frac{1}{2}I_2 \oplus 0, \quad A_{j+1} = \frac{1}{2m}I_2 \oplus v_j v_j^*, \quad 1 \leq j \leq m.$$

The identities $\|v_j\|^2 = 1 - 1/m$ and $\sum_j v_j v_j^* = I_W$ imply

$$A_i \succeq 0, \quad \text{tr } A_i = 1, \quad \sum_i A_i = I_n.$$

Consequently $f_n(x) = \det(\sum_i x_i A_i)$ has nonnegative coefficients and $f_n(1^n) = 1$. It is real stable: if every $\text{Im } x_i > 0$, then $\text{Im}(\sum_i x_i A_i) \succ 0$, so the matrix is invertible.

The Gram matrix of the vectors v_j is $I_m - \mathbf{1}\mathbf{1}^*/m$, whose principal minors of order $m - 1$ all equal $1/m$. Cauchy–Binet therefore gives

$$f_n(x) = \frac{1}{4} \left(x_1 + \frac{x_2 + \cdots + x_n}{m} \right)^2 \frac{e_{m-1}(x_2, \dots, x_n)}{m}.$$

The final factor omits one uniformly chosen variable among $x_2, \dots, x_n$. The three terms x_1^2 , $2x_1(x_2 + \cdots + x_n)/m$, and $((x_2 + \cdots + x_n)/m)^2$ have coefficient masses $1/4$, $1/2$, and $1/4$. The middle term fills the omitted variable with probability $1/m$; the last fills it with probability $2/m - 1/m^2$. Hence the masses of support deficits $0, 1, 2$ are

$$p_0 = \frac{1}{2m}, \quad p_1 = \frac{3}{4} - \frac{1}{4m^2}, \quad p_2 = \frac{(m-1)^2}{4m^2}.$$

Their signed score is

$$\mathcal{L}_n(f_n) = p_0 - \frac{p_1}{n} + \frac{p_2}{\binom{n}{2}} = \frac{-m^3 + 4m^2 - 3m + 2}{4m^3(m+1)} = -\frac{1}{4n} + O(n^{-2}).$$

For $n \geq 5$, its numerator equals $-m^2(m-4) - 3m + 2 < 0$.

To compare this score with a map invariant, let $G_n(X) = \sum_i X_{ii} A_i$. This map is completely positive and bistochastic: its Choi matrix is $\sum_i E_{ii} \otimes A_i \succeq 0$, and the balance identities give its two marginals. Its determinant depends only on diagonal entries, so only the identity permutation survives in $\det(\partial_X)$ and

$$\Phi_n(G_n) = \frac{[x_1 \cdots x_n] f_n}{n!} = \frac{1}{2(n-1)n!} > 0.$$

The balanced stable tuple conditions therefore cannot imply $\mathcal{L}_n(f) \geq -\varepsilon_n/n$ for a sequence $\varepsilon_n \rightarrow 0$. Any proof of that scale of lower bound for Φ_n must use additional information about the tuples arising from a common map.

S5. STRUCTURE OF THE MAP AND THE CONJECTURED ORDER

Theorem 4.8 excludes certifying the order of Conjecture 6.1 with the recursive coefficient programs. An entanglement-breaking submap supplies additional information through its rank-one expansion, which retains the weights of complementary compressions.

S5.1. Entanglement-breaking submaps. A map S is entanglement breaking if it has a finite representation $S(X) = \sum_a (b_a^* X b_a) a_a a_a^*$. For $H \in \mathfrak{B}_n$, define

$$(S5.1) \quad b_{\text{EB}}(H) = \frac{1}{n} \max \{ \text{tr } S(I) : S \text{ entanglement breaking, } H - S \text{ positive} \}.$$

The feasible set is compact: $0 \preceq S(I) \preceq I$ bounds the norm of S , and the two cones are closed. Hence the maximum exists and lies in $[0, 1]$. A decomposition $H = \beta E + (1 - \beta)B$ with $E, B \in \mathfrak{B}_n$ and E entanglement breaking gives $b_{\text{EB}}(H) \geq \beta$, but the definition does not require proportional marginals for S .

Proposition S5.1 (A bound from an entanglement-breaking submap). *Let $H \in \mathfrak{B}_n$, let S be entanglement breaking, and suppose that $R = H - S$ is positive. Put $T = R(I)$, and let J be the Poisson-binomial random variable with success probabilities the eigenvalues of T , a sum of independent Bernoulli variables. If $\gamma_j \leq 0$ is a lower bound for Φ_j on $\mathfrak{B}_j$ for each $j \geq d_*$, then*

$$(S5.2) \quad \Phi_n(H) \geq \frac{1}{n!} \sum_{j=d_*}^n j! \gamma_j \Pr(J = j).$$

Consequently, with $\eta = 1 - b_{\text{EB}}(H)$,

$$(S5.3) \quad \Phi_n(H) \geq - \sum_{j=d_*}^n \frac{\eta^j (1 - \eta)^{n-j}}{j(n-j)!}.$$

The sums are empty when $n < d_*$.

Proof. Choose a finite representation

$$S(X) = \sum_{a=1}^N (b_a^* X b_a) a_a a_a^*, \quad \|b_a\| = 1.$$

Input normalization is absorbed into the output vectors, so $\sum_a a_a a_a^* = S(I) = I - T$. For a subset $A \subseteq [N]$ of cardinality m , let U_A, V_A be the matrices with columns a_a, b_a , respectively, and let P_A, Q_A be isometries onto their respective orthogonal complements. Rank-deficient subsets have zero Gram weight and can be discarded. Expanding the output determinant along the rank-one terms of S gives the exact identity

$$(S5.4) \quad \Gamma_n(H) = \sum_{m=0}^n \sum_{|A|=m} \det(U_A^* U_A) \det(V_A^* V_A) \Gamma_{n-m}(R_A), \quad R_A(Z) = P_A^* R(Q_A Z Q_A^*) P_A,$$

with $\Gamma_0 = 1$.

To verify the identity and its normalization, introduce distinct parameters in the rank-one sum for S . The output determinant coefficient indexed by A is

$$\det(U_A^* U_A) \det(P_A^* R(X) P_A) \prod_{a \in A} (b_a^* X b_a).$$

For a homogeneous polynomial f of degree $n - m$, the input determinant differential satisfies

$$\det(\partial_X) \left[f(X) \prod_{a \in A} (b_a^* X b_a) \right] = \det(V_A^* V_A) \det(\partial_Z) f(Q_A Z Q_A^*).$$

In a unitary basis adapted to the input vectors, the derivatives that hit the m linear factors exhaust their rows and columns; their alternating sum is the Gram determinant. The remaining derivatives act on the complementary block, which proves (S5.4) without an additional factorial.

Every R_A is positive. Its invariant is nonnegative in dimensions $j = n - m < d_*$ by the definition of d_* and Lemma 2.2. For $j \geq d_*$, the same lemma gives

$$\Gamma_j(R_A) \geq j! \gamma_j \det R_A(I) \geq j! \gamma_j \det(P_A^* T P_A),$$

because $R_A(I) \preceq P_A^* R(I) P_A = P_A^* T P_A$ and $\gamma_j \leq 0$. Hadamard's inequality gives $\det(V_A^* V_A) \leq 1$. Dropping this factor from the lower bound and summing over all subsets of cardinality m gives

$$\sum_{|A|=m} \det(U_A^* U_A) \det(V_A^* V_A) \Gamma_j(R_A) \geq j! \gamma_j [t^m] \det(T + t(I - T)).$$

Subsets with dependent input vectors can be restored in the lower bound: their contribution is zero, whereas the restored output-Gram term multiplied by $\gamma_j \leq 0$ is nonpositive. The coefficient identity is the rank-one expansion of the output determinant. Since $0 \preceq T \preceq I$, diagonalization gives

$$[t^{n-j}] \det(T + t(I - T)) = \Pr(J = j).$$

Divide by $n!$ to obtain (S5.2).

For the final assertion, use $\gamma_j = -1/j$ and define

$$f(j) = \begin{cases} 0, & j < d_*, \\ (j-1)!, & j \geq d_*. \end{cases}$$

The sequence f has nonnegative second differences. At the two indices that cross the cutoff, these differences are $(d_* - 1)!$ and $(d_* - 2)(d_* - 1)!$; above the cutoff they are $(j-1)!(j^2 - j + 1)$. If two Bernoulli probabilities are p, q , condition on the sum k of the remaining variables. Their contribution to the expectation is

$$f(k) + (p+q)(f(k+1) - f(k)) + pq(f(k+2) - 2f(k+1) + f(k)).$$

Replacing p, q by their average keeps their sum fixed and increases their product, so it cannot decrease the expectation. Consequently the expectation is largest when all probabilities equal their mean. For a direct compactness argument, among the maximizers with a fixed sum choose one minimizing the sum of squared probabilities. Averaging any unequal pair contradicts that minimality. It follows that

$$\mathbb{E}f(J) \leq \mathbb{E}f(\text{Bin}(n, \bar{t})), \quad \bar{t} = \frac{\text{tr } T}{n} = 1 - \frac{\text{tr } S(I)}{n}.$$

Choose S attaining (S5.1) and use $(j-1)!\binom{n}{j}/n! = 1/[j(n-j)!]$ to obtain (S5.3). $\square$

The tail bound (S5.3) decays exponentially in $n b_{\text{EB}}(H)$, so a large entanglement-breaking submap forces the conjectured rate.

Corollary S5.2. *For every $H \in \mathfrak{B}_n$,*

$$(S5.5) \quad \Phi_n(H) \geq -\frac{e(1+1/d_*)}{n} \exp[-n \min\{b_{\text{EB}}(H), 1/2\}].$$

If $b_{\text{EB}}(H_n) \geq (d_* - 1) \log n/n$, then for all sufficiently large n ,

$$(S5.6) \quad \Phi_n(H_n) \geq -\frac{e(1+1/d_*)}{n^{d_*}} \geq -\frac{5e}{4(n)_{d_*}}.$$

Proof. For $0 < \eta \leq 1$, put $x = (1 - \eta)/\eta$. Since $n/(n - m) \leq 1 + m/d_*$ when $m \leq n - d_*$,

$$(S5.7) \quad \sum_{j=d_*}^n \frac{\eta^j (1 - \eta)^{n-j}}{j(n-j)!} = \frac{\eta^n}{n} \sum_{m=0}^{n-d_*} \frac{x^m}{m!} \frac{n}{n-m} \leq \frac{\eta^n}{n} \left(1 + \frac{x}{d_*}\right) e^x.$$

Scale an optimizing entanglement-breaking submap down to normalized trace $\beta = \min\{b_{\text{EB}}(H), 1/2\}$. Apply the proof of Proposition S5.1 with $\eta = 1 - \beta \geq 1/2$. Then $x \leq 1$ and $\eta^n \leq e^{-n\beta}$ give (S5.5). If $n < d_*$, the conclusion instead follows directly from $\Phi_n(H) \geq 0$. For large n , the assumed threshold is at most $1/2$ and gives the first inequality in (S5.6). The second uses $d_* \geq 4$ and $(n)_{d_*} \leq n^{d_*}$. $\square$

Section S6.2 gives negative maps satisfying this criterion. For arbitrary maps with negative invariant, the required lower bound on $b_{\text{EB}}(H)$ remains unproved.

S5.2. Coherent extensions and the noncentral obstruction. Pinched padding discards the off-diagonal blocks. Retaining those blocks tests whether coupling to the added dimensions can produce a larger negative part. Write $\text{Ad}_K(X) = KXK^*$. Fix a core $A \in \mathfrak{B}_r$ and a coupling $K \in M_r$ such that $B = A - \text{Ad}_K$ is positive. The pair (A, K) is called admissible, and its coherent extension is

$$T_n \begin{pmatrix} X & Z \\ W & Y \end{pmatrix} = \begin{pmatrix} A(X) & KZ \\ WK^* & Y \end{pmatrix} \quad (n \geq r).$$

The decomposition $T_n = \text{Ad}_{K \oplus I_{n-r}} + B_{\text{corner}}$ proves positivity, and the diagonal blocks give bistochasticity. In the asymptotic statements, r is fixed but A and K may depend on n .

Put $S = KK^*$, $\beta = \|B\|_{\text{HS} \rightarrow \text{HS}}$, and $c_j(B, K) = (r)_j [z^j] \Phi_r(\text{Ad}_K + zB)$ for $0 \leq j \leq r$. Admissibility gives $S \preceq I$ and $0 \leq \beta \leq 1$; the contraction argument is recorded in Section S5.4. The corner identity (C.5) gives

$$(S5.8) \quad \Phi_n(T_n) = \sum_{j=0}^r \frac{c_j(B, K)}{(n)_j}, \quad c_0(B, K) = \det S.$$

The same coefficient definition makes sense for any Hermiticity-preserving B and any $Q \in M_r$. Define

$$\mathcal{W}_B(Q) = \sum_{j=0}^r c_j(B, Q).$$

The function $\mathcal{W}_B$ is a real polynomial in the real and imaginary parts of Q , with $\mathcal{W}_B(0) = \Gamma_r(B)$.

Proposition S5.3 (Gamma transform). *For the residual B and coupling K of an admissible pair,*

$$(S5.9) \quad \Phi_n(T_n) = \frac{1}{(n)_r} \mathbb{E} \mathcal{W}_B(\sqrt{G_n} K),$$

where G_n is a scalar Gamma random variable with density $e^{-t} t^{n-r} / (n-r)!$ on $t > 0$.

Proof. Homogeneity gives $c_j(B, \sqrt{t}K) = t^{r-j} c_j(B, K)$, and $\mathbb{E} G_n^{r-j} = (n-j)! / (n-r)!$. Substitution in (S5.9) gives (S5.8) term by term. $\square$

The Gamma measure is positive, but the noncentral polynomial need not have positive coefficients.

Proposition S5.4 (Lifting a negative noncentral value). *Let $B \in \mathfrak{B}_d$ and $Q \in M_d$. If $\mathcal{W}_B(Q) < 0$, there is an explicit admissible d -core family with*

$$\Phi_n(T_n) = n^{-d} (\mathcal{W}_B(Q) + O_{B,Q,d}(n^{-1})).$$

Consequently, a universal estimate $-\mu_n = O(n^{-d_*})$ would require $\mathcal{W}_B(Q) \geq 0$ for every $B \in \mathfrak{B}_d$, every $d < d_*$, and every $Q \in M_d$.

Proof. Choose $c \geq d \|Q\|_{\text{op}}^2$ and, for $n \geq \max(d, c)$, set

$$K_n = n^{-1/2} Q, \quad A_n = (1 - c/n)B + (c/n)\Delta_d.$$

The map A_n is bistochastic. Its residual is positive because

$$A_n - \text{Ad}_{K_n} = (1 - c/n)B + n^{-1}(c\Delta_d - \text{Ad}_Q) \geq 0,$$

where $QXQ^* \preceq \|Q\|_{\text{op}}^2 \text{tr}(X)I$ for $X \succeq 0$. Write this residual as $B_n = B + O(n^{-1})$. The exact coefficient formula and homogeneity give

$$n^d \Phi_n(T_n) = \sum_{j=0}^d \frac{n^j}{(n)_j} c_j(B_n, Q) = \mathcal{W}_B(Q) + O(n^{-1}).$$

Each c_j depends polynomially on the residual, which justifies the error term. A negative value in dimension $d < d_*$ would therefore contradict the proposed $O(n^{-d_*})$ upper bound for $-\mu_n$. $\square$

The definition of d_* gives this sign only at $Q = 0$. It does not settle the nonzero shifts, nor has the sign condition alone been shown sufficient to control every degeneration of an admissible coupling.

S5.3. Critical directions near a reduction compression. General coefficient estimates prove the conjectured rate when d_* coupling singular values are $O(n^{-1/2})$, or when $d_* - 1$ are $O(n^{-1})$; see Theorem S5.10 and Proposition S5.13. At the critical $n^{-1/2}$ scale, the next theorem instead uses the structure of the residual compression. For odd $q \geq 3$, write

$$(S5.10) \quad \mathcal{R}_q(X) = \frac{\text{tr}(X)I_q - X^\top}{q-1}.$$

We use $\|\cdot\|_F$ for the Frobenius norm and $x_+ = \max\{x, 0\}$ for the positive part.

Theorem S5.5 (Critical directions near a reduction compression). *Fix an odd integer $q \geq 3$, an integer $h \geq 1$, and $\alpha_0, L > 0$, and put $r = h + q$. For each $n \geq r$, let (A_n, K_n) be an admissible coherent pair. Suppose that, in compatible input and output coordinates,*

$$K_n = H_n \oplus R_n, \quad H_n \in M_h, \quad \|R_n\|_F \leq Ln^{-1/2}.$$

Call H_n the hard block and R_n the soft block. Let $B_n = A_n - \text{Ad}_{K_n}$. Its soft compression $(B_n)_{00}$ is the restriction to the last q input and output coordinates. Suppose it satisfies, for some $\alpha_n \geq \alpha_0$,

$$\delta_n := \|(B_n)_{00} - \alpha_n \mathcal{R}_q\|_{\text{HS} \rightarrow \text{HS}} \leq 1.$$

There is a constant $C_{r, \alpha_0, L}$ such that

$$(S5.11) \quad (-\Phi_n(T_n))_+ \leq C_{r, \alpha_0, L} \left(\frac{\delta_n}{(n)_q} + \frac{1}{(n)_{q+1}} \right) \quad (n \geq r).$$

In particular, if $\delta_n \leq M/n$, then the negative part is $O_{r, \alpha_0, L, M}(n^{-q-1})$. No singular value is required to vanish. The hard spectrum is arbitrary subject to admissibility. When $q+1 \geq d_*$, this proves an $O(n^{-d_*})$ bound for this class.

Proof mechanism. The noncentral polynomial of $\alpha \mathcal{R}_q$ is the expectation of a squared determinant under a skew-symmetric Gaussian shift. For odd q , its zero set consists exactly of skew-symmetric shifts, and Lemma S5.15 bounds it below by a positive multiple of $\|(Q + Q^\top)/2\|_F^2$. Positivity of the full residual controls its cross terms with the hard directions. A compression error δ introduces errors of size $O(\sqrt{\delta})$ in these terms. Completing the square against the reference coercivity gives the pointwise estimate (S5.23). The Gamma transform then bounds the error term by $C\delta/(n)_q$ and the remaining terms by $C/(n)_{q+1}$. Section S5.4.3 proves these estimates uniformly over the stated admissible pairs. $\square$

For coherent extensions, control of general positive compressions at this critical scale remains open. For arbitrary maps, a concentration estimate restricts any sequence whose negative invariant is of order $1/n$. Define

$$q(H) = \|H\|_{\text{HS}, \text{super}}^2 := \sum_{i,j=1}^n \|H(E_{ij})\|_{\text{HS}}^2 = \|C_H\|_F^2.$$

Section S6.3 proves, with $q = q(H)$,

$$(S5.12) \quad \Phi_n(H) \geq -\frac{3}{n} \exp\left(-\frac{n^2 - q}{64(n+1)}\right).$$

Thus $\Phi_n(H_n) \leq -c/n$, for fixed $c > 0$, requires $q(H_n) = n^2 - O(n)$. Section S6 places two further restrictions beside this one: a contraction-radius bound (Theorem S6.2), which forces every sequence with polynomially negative invariant to act as a near-isometry on traceless matrices, and the first-order conditions at a negative global minimizer, whose cofactor is $\mu_n I$ plus a separable slack (Theorem S6.8). The universal n^{-d_*} bound and the noncentral sign problem remain open.

S5.4. Uniform spectral control of coherent extensions. Use the admissible pair (A, K) , residual B , and coefficients c_j from Section 6, with $r \leq n$. The pair may vary with n . Admissibility gives $B(I) = I - KK^* \preceq I$ and $B^*(I) = I - K^*K \preceq I$. The spectral-compression argument in Lemma A.1, with a contraction in place of the isometry, gives $B(X)^2 \preceq B(X^2)$ for Hermitian X . Taking traces and using the real/imaginary decomposition proves $\beta \leq 1$.

Lemma S5.6 (Exterior-power coefficient bound). *For every admissible pair and $0 \leq j \leq r$,*

$$(S5.13) \quad |c_j(B, K)| \leq (r)_j \beta^j e_{r-j}(S).$$

For $j = 0$, the factor β^0 is read as one.

Proof. The case $j = 0$ is exact. For $j \geq 1$, put $D_j = \binom{r}{j}$ and let $V_j : \bigwedge^j \mathbb{C}^r \rightarrow (\mathbb{C}^r)^{\otimes j}$ be the antisymmetric isometry, normalized by $1/\sqrt{j!}$. On M_{D_j} define

$$\widehat{B}_j(X) = V_j^* B^{\otimes j} (V_j X V_j^*) V_j.$$

This map need not be positive. The relevant bound is its HS operator norm, which is at most β^j . Its Choi matrix consequently satisfies

$$\|C_{\widehat{B}_j}\|_{\text{op}} \leq \|C_{\widehat{B}_j}\|_F = \|\widehat{B}_j\|_{\text{HS,super}} \leq D_j \beta^j.$$

To identify the coefficient pairing, use the formal circular Wick functional Λ_B with covariance $\Lambda_B(Z_{ai} \overline{Z}_{bj}) = B(E_{ij})_{ab}$: balanced higher moments are sums over all pairings, and unbalanced moments are zero. This is an algebraic functional; the Choi covariance may be indefinite, so no positive probability measure is asserted. Set $p_j(Z, K) = [u^{r-j}] \det(uK + Z)$. Determinant expansion and the scalar Gaussian moment $\mathbb{E}|g|^{2(r-j)} = (r-j)!$ give

$$\Gamma_r(\text{Ad}_K + zB) = \sum_{j=0}^r (r-j)! z^j \Lambda_B(|p_j(Z, K)|^2).$$

Thus the functional in the j th term is $c_j(B, K)$. The identity can also be checked for positive definite Choi covariance by ordinary Wick expansion and extended polynomially to all Hermitian Choi matrices.

Expand p_j in the $j \times j$ minors of Z . Its coefficient vector consists of the signed complementary minors of K , whose squared norm is

$$\sum_{|I|=|J|=j} |\det K_{I^c, J^c}|^2 = e_{r-j}(S).$$

For j -element subsets I, J, I', J' , antisymmetrizing the covariance gives

$$\Lambda_B(\det Z_{I,J} \overline{\det Z_{I',J'}}) = j! (\widehat{B}_j(E_{J,J'}))_{I,I'}.$$

The matrix units here are indexed by subsets in the exterior basis. Thus the matrix of Wick pairings is $j! C_{\widehat{B}_j}$ when its indices are ordered as (J, I) and (J', I') . The factor $j!$ counts the Wick pairings. Cauchy–Binet and the preceding Choi norm bound therefore give $|c_j| \leq j! D_j \beta^j e_{r-j}(S)$, which is (S5.13). $\square$

Theorem S5.7 (Spectral envelope and sign cutoff). *Every admissible pair satisfies, uniformly for $n \geq r$,*

$$(S5.14) \quad |\Phi_n(T_n)| \leq \det \left(S + \frac{r\beta}{n} I_r \right),$$

$$(S5.15) \quad \Phi_n(T_n) \geq 2 \det S - \det \left(S + \frac{r\beta}{n} I_r \right).$$

If $\Phi_n(T_n) < 0$, then

$$(S5.16) \quad \sigma_{\min}(K)^2 < \frac{L_r \beta}{n}, \quad L_r = \frac{r}{2^{1/r} - 1},$$

where $\sigma_{\min}$ is the smallest singular value.

Proof. For $n \geq r$, $(r)_j/(n)_j \leq (r/n)^j$. Sum (S5.13) in (S5.8) and use $\sum_j x^j e_{r-j}(S) = \det(S + xI)$ to obtain (S5.14). Retaining $c_0 = \det S$ exactly proves (S5.15).

If K is invertible and the opposite of (S5.16) holds, then

$$\det \left(S + \frac{r\beta}{n} I \right) \leq \det S \left(1 + \frac{r\beta}{n\sigma_{\min}(K)^2} \right)^r \leq 2 \det S,$$

so the invariant is nonnegative. For singular K , the cutoff holds whenever $\beta > 0$. If $\beta = 0$, then $B = 0$ and bistochasticity makes K unitary, giving $\Phi_n(T_n) = 1$. $\square$

The sign cutoff forces one small coupling direction. A sharper estimate uses positivity to combine the constant and linear terms before bounding the higher coefficients. For a real number x , write $x_+ = \max(x, 0)$.

Proposition S5.8 (Uniform negative-part bound). *For $n \geq r \geq 2$, every admissible pair satisfies*

$$(S5.17) \quad (-\Phi_n(T_n))_+ \leq \frac{(r^2 + r/2)\beta^2}{n^2} e_{r-2} \left(S + \frac{r\beta}{n} I_r \right).$$

In particular, the negative part is $O_r(n^{-2})$, uniformly over pairs that may vary with n .

Proof. Unitary changes of input and output reduce to $K = \text{diag}(k_1, \dots, k_r)$ with $k_i \geq 0$, preserving the invariant and β . Put $P = \prod_i k_i$, $s_i = \prod_{a \neq i} k_a$, and $b_{ih} = B(E_{hh})_{ii} \geq 0$. Cofactor expansion gives $c_1 = \sum_{i,j} s_i s_j B(E_{ij})_{ij}$. For $x_i = \sqrt{s_i} e^{i\theta_i}$, independent uniform phases give

$$0 \leq \mathbb{E}_\theta x^* B(xx^*) x = c_1 + \sum_{i \neq h} b_{ih} s_i s_h.$$

Since $s_i s_h = P \prod_{a \neq i,h} k_a$, define $R = \sum_{i \neq h} b_{ih} \prod_{a \neq i,h} k_a$ and obtain $c_1 \geq -PR$. The bounds $\sum_{i,h} b_{ih} = \text{tr } B(I) \leq r\beta$ and $b_{ih} \leq \beta$ give, by weighted Cauchy–Schwarz, $R^2 \leq 2r\beta^2 e_{r-2}(S)$. Consequently, the first two terms satisfy

$$c_0 + \frac{c_1}{n} \geq P^2 - \frac{PR}{n} \geq -\frac{r\beta^2}{2n^2} e_{r-2}(S).$$

Put $x = r\beta/n$. The absolute tail is at most

$$\sum_{j=2}^r x^j e_{r-j}(S) \leq x^2 e_{r-2}(S + xI).$$

Indeed, the coefficient of $x^{j-2} e_{r-j}(S)$ in the last elementary symmetric polynomial is $\binom{j}{2} \geq 1$. Combining these bounds proves (S5.17). Since $S \preceq I$ and $\beta \leq 1$, its right-hand side is bounded by a constant depending only on r times n^{-2} . $\square$

Corollary S5.9 (Growing cores). *For every admissible pair with $n \geq r \geq 2$,*

$$(-\Phi_n(T_n))_+ \leq \frac{(r^2 + r/2)\binom{r}{2}}{n^2} \left(1 + \frac{r}{n} \right)^{r-2}.$$

If $n \geq r^2$, then

$$(-\Phi_n(T_n))_+ \leq \frac{e r^4}{2n^2}.$$

Consequently, if $r = r(n) = o(n^{1/4})$, then $(-\Phi_n(T_n))_+ = o(1/n)$ uniformly over all admissible core maps and couplings.

Proof. Since $S \preceq I_r$ and $\beta \leq 1$,

$$e_{r-2} \left(S + \frac{r\beta}{n} I_r \right) \leq \binom{r}{2} \left(1 + \frac{r}{n} \right)^{r-2}.$$

Proposition S5.8 proves the first bound. For the second, use

$$(r^2 + r/2) \binom{r}{2} = \frac{r^4}{2} \left(1 - \frac{1}{2r} - \frac{1}{2r^2} \right) \leq \frac{r^4}{2},$$

and, when $n \geq r^2$,

$$\left(1 + \frac{r}{n} \right)^{r-2} \leq \exp \left(\frac{r(r-2)}{n} \right) \leq e.$$

Finally, $r = o(n^{1/4})$ implies $n \geq r^2$ eventually and $r^4/n^2 = o(1/n)$. $\square$

S5.4.1. Matching orders for coherent extensions under spectral restrictions. The least negative dimension d_* , defined in (6.1), lets us compare two spectral restrictions on the coupling. The first has the order predicted by Conjecture 6.1; the second can have a faster decay.

Admissible pairs form a compact set: A ranges over the compact class $\mathfrak{B}_r$ and $\|K\|_{\text{op}} \leq 1$. The spectral restrictions below are closed, so the stated minima exist.

Theorem S5.10. *Fix $r \geq d_*$. Order the singular values decreasingly. For fixed $L \geq 0$, let $m_{r,L}^{\text{soft}}(n)$ be the minimum of $\Phi_n(T_n)$ over admissible pairs with $\sigma_{r-d_*+1}(K)^2 \leq L/n$. Then, as $n \rightarrow \infty$,*

$$-m_{r,L}^{\text{soft}}(n) = \Theta_{r,L}((n)_{d_*}^{-1}).$$

For fixed $\kappa \geq 1$, let $m_{r,\kappa}^{\text{cond}}(n)$ be the minimum over admissible pairs with $\sigma_{\max}(K) \leq \kappa \sigma_{\min}(K)$, including $K = 0$. Then

$$-m_{r,\kappa}^{\text{cond}}(n) = \Theta_{r,\kappa}((n)_r^{-1}).$$

Proof. In the first class at least d_* singular values have square at most L/n , and all others have square at most one. The envelope gives

$$|\Phi_n(T_n)| \leq (1 + r/n)^{r-d_*} ((L + r)/n)^{d_*} = O_{r,L}(n^{-d_*}).$$

For the matching negative value, choose $H \in \mathfrak{B}_{d_*}$ with $\Phi_{d_*}(H) < 0$ and take $A = H \boxplus \text{Id}_{r-d_*}$, $K = 0_{d_*} \oplus I_{r-d_*}$. Its ambient extension is $H \boxplus \text{Id}_{n-d_*}$, so

$$-\Phi_n(T_n) = \frac{-d_*! \Phi_{d_*}(H)}{(n)_{d_*}}.$$

In the second class, a negative value with $K \neq 0$ obeys $n \sigma_{\max}(K)^2 < \kappa^2 L_r \beta$ by (S5.16). The envelope yields

$$-\Phi_n(T_n) \leq \frac{\beta^r (r + \kappa^2 L_r)^r}{n^r} \leq \frac{(r + \kappa^2 L_r)^r}{n^r}.$$

The same estimate covers $K = 0$. A negative map in $\mathfrak{B}_r$ exists by pinching a negative d_* -dimensional map; taking it as A with $K = 0$ gives the matching order $(n)_r^{-1}$. $\square$

For $r = d_*$ the two exponents agree. For $r > d_*$ the condition-number restriction excludes the partially zero coupling used in the first class, so it yields exponent r . Neither restriction controls every admissible pair with fewer than d_* small singular values and unbounded condition number.

The sign of the first surviving compression gives a stronger bound for singular couplings.

Proposition S5.11 (A rank restriction on coherent extensions). *Fix $r \geq d_*$ and let (A, K) be an admissible coherent pair. Write $B = A - \text{Ad}_K$, $S = KK^*$, and $\beta = \|B\|_{\text{HS} \rightarrow \text{HS}}$. If*

$$\dim \ker K \geq d_* - 1,$$

then, for every $n \geq r$,

$$(-\Phi_n(T_n))_+ \leq \left(\frac{r\beta}{n}\right)^{d_*} e_{r-d_*} \left(S + \frac{r\beta}{n} I_r\right).$$

In particular, the negative part is $O_r(n^{-d_*})$, uniformly over admissible pairs that may depend on n . The minimum over this class has negative part $\Theta_r((n)^{-1})$.

Proof. Put $q = \dim \ker K$. The determinant polynomial $\det(uK + Z)$ has degree at most $r - q$ in u . The coefficient representation in Lemma S5.6 therefore gives $c_j(B, K) = 0$ for $j < q$. Unitary input and output changes reduce K to $D \oplus 0_q$, with D an invertible $(r - q)$ -dimensional diagonal matrix. The term of highest degree in u in $\det(uK + Z)$ is

$$u^{r-q} \det D \det Z_{00},$$

where Z_{00} denotes the block on the input and output kernels of K . If B_{00} is the corresponding positive compression of B , the formal Wick identity consequently gives

$$c_q(B, K) = |\det D|^2 \Gamma_q(B_{00}).$$

When $q = d_* - 1$, Lemma 2.2 with zero lower seed and the definition of d_* imply $\Gamma_q(B_{00}) \geq 0$. When $q \geq d_*$, all coefficients below d_* vanish. Either way, every contribution below d_* is nonnegative.

The exterior-power coefficient estimate and $(r)_j/(n)_j \leq (r/n)^j$ now give, with $x = r\beta/n$,

$$(-\Phi_n(T_n))_+ \leq \sum_{j=d_*}^r x^j e_{r-j}(S) \leq x^{d_*} e_{r-d_*}(S + xI_r).$$

For the final inequality, the coefficient of $x^{j-d_*} e_{r-j}(S)$ in the last elementary symmetric polynomial is $\binom{j}{d_*} \geq 1$. Since $S \preceq I_r$ and $\beta \leq 1$, the right-hand side is at most

$$\frac{r^{d_*}}{n^{d_*}} \binom{r}{d_*} \left(1 + \frac{r}{n}\right)^{r-d_*}.$$

For the matching order, choose a negative map $H_* \in \mathfrak{B}_{d_*}$ and take $A = H_* \boxplus \text{Id}_{r-d_*}$ and $K = 0_{d_*} \oplus I_{r-d_*}$. This pair belongs to the stated class, and its extension has

$$-\Phi_n(T_n) = \frac{-d_*! \Phi_{d_*}(H_*)}{(n)_{d_*}}.$$

This proves the sharp order. $\square$

The sign argument permits $d_* - 1$ exact zero singular values; the remaining singular values need not be small or comparable. Exact zeros remove the lower coefficient terms; for small nonzero singular values, those terms must also be bounded.

Corollary S5.12 (Growing cores under the rank restriction). *Under the preceding rank hypothesis, if $n \geq r^2$, then*

$$(-\Phi_n(T_n))_+ \leq \frac{e r^{2d_*}}{d_*! n^{d_*}}.$$

Consequently the negative part is $o(1/n)$ whenever $r = o(n^{(d_*-1)/(2d_*)})$.

Proof. Use $\binom{r}{d_*} \leq r^{d_*}/d_*!$ and $(1 + r/n)^{r-d_*} \leq \exp(r^2/n) \leq e$ in the preceding estimate. The stated growth condition implies $r^2 = o(n)$ and $r^{2d_*} = o(n^{d_*-1})$. $\square$

Proposition S5.13 (Small nonzero coupling directions). *Let $r \geq d_*$, put $q = d_* - 1$ and $h = r - q$, and let (A, K) be an admissible coherent pair. Order the eigenvalues of $S = KK^*$ as $s_1 \geq \dots \geq s_r \geq 0$, and set*

$$P = \prod_{i=1}^h s_i, \quad D = e_h(S) - P, \quad x = \frac{r\beta}{n}.$$

For every $n \geq r$,

$$(S5.18) \quad (-\Phi_n(T_n))_+ \leq \sum_{j=1}^{q-1} x^j e_{r-j}(S) + x^q (2\sqrt{PD} + D) + x^{q+1} e_{r-q-1}(S + xI_r).$$

Consequently, if the q smallest singular values of K are at most L/n , then $(-\Phi_n(T_n))_+ = O_{r,L}(n^{-d_*})$, uniformly over admissible pairs that may depend on n .

Proof. After unitary changes of input and output coordinates, take $K = \text{diag}(\sqrt{s_1}, \dots, \sqrt{s_r})$ and let $K_0 = \text{diag}(\sqrt{s_1}, \dots, \sqrt{s_h}, 0, \dots, 0)$. The residual B is kept fixed. The pair formed by $B + \text{Ad}_{K_0}$ and K_0 need not be bistochastic; neither the coefficient identity nor the following sign argument requires that normalization. The leading-coefficient argument at a rank-deficient coupling gives

$$c_q(B, K_0) = P \Gamma_q(B_{00}) \geq 0,$$

where B_{00} is the positive compression to the last q input and output coordinates. The inequality follows from $q < d_*$ and the normalization lemma with zero seed. This also holds if $P = 0$.

In the proof of the exterior-power coefficient bound, write $c_q(B, K) = v(K)^* M_q(B) v(K)$, where $v(K)$ is the vector of signed complementary h -minors and $\|M_q(B)\|_{\text{op}} \leq (r)_q \beta^q$. For diagonal K , its only nonzero minors are principal. The vector $v(K_0)$ has one possibly nonzero coordinate, which agrees with the corresponding coordinate of $v(K)$. Thus

$$\|v(K_0)\|^2 = P, \quad \|v(K) - v(K_0)\|^2 = D.$$

The quadratic-form difference is bounded by

$$|c_q(B, K) - c_q(B, K_0)| \leq (r)_q \beta^q (2\sqrt{PD} + D).$$

For $1 \leq j < q$ and $j > q$, use the usual exterior-power estimate. Discard $c_0 = \det S \geq 0$, apply $(r)_j / (n)_j \leq (r/n)^j$, and bound the upper-index tail by

$$\sum_{j=q+1}^r x^j e_{r-j}(S) \leq x^{q+1} e_{r-q-1}(S + xI_r).$$

This proves (S5.18).

For the stated consequence put $\tau = \max_{i>h} \sqrt{s_i}$. Since $S \preceq I_r$, every product contributing to $e_{r-j}(S)$ for $j < q$ includes at least $q - j$ soft factors, and hence

$$e_{r-j}(S) \leq \binom{r}{j} \tau^{2(q-j)}, \quad D \leq \left(\binom{r}{h} - 1 \right) \tau^2, \quad P \leq 1.$$

With $\tau \leq L/n$, the term of index $j < q$ is $O_{r,L}(n^{-(2q-j)}) = O_{r,L}(n^{-q-1})$, the middle term is $O_{r,L}(n^{-q-1})$, and the last term has the same bound. $\square$

This estimate uses scale n^{-1} for singular values, or n^{-2} for the corresponding eigenvalues of KK^* . It does not establish the desired conclusion at the critical singular-value scale $n^{-1/2}$.

S5.4.2. A noncentral determinant problem at the critical scale. The Gamma representation from Section 6 does not imply coefficientwise positivity. For example, the reduction map $B(X) = \text{tr}(X)I_2 - X$ has $c_j(B, I_2) = (1, -2, 2)$ for $j = 0, 1, 2$, and $\mathcal{W}_B(tI_2) = |t|^4 - 2|t|^2 + 2$. This polynomial is positive.

Central nonnegativity controls the first two coefficients; the difference begins at fourth order.

Proposition S5.14 (A fourth-order remainder below d_*). *Let $B : M_q \rightarrow M_q$ be positive, with $q < d_*$, and put $\beta = \|B\|_{\text{HS} \rightarrow \text{HS}}$. For every $Q \in M_q$,*

$$(S5.19) \quad \mathcal{W}_B(Q) \geq -q! \sum_{k=2}^q \left(1 - \frac{1}{k!}\right) \beta^{q-k} e_k(QQ^*).$$

In particular, $(-\mathcal{W}_B(Q))_+ = O_{q,\beta}(\|Q\|^4)$ near $Q = 0$. For an admissible coherent core of dimension $q < d_*$,

$$(S5.20) \quad (-\Phi_n(T_n))_+ \leq q! \sum_{k=2}^q \left(1 - \frac{1}{k!}\right) \frac{\beta^{q-k} e_k(KK^*)}{(n)_{q-k}}.$$

Thus, when $q = d_* - 1$ and all singular values of K are at most $Ln^{-3/4}$, the negative part is $O_{q,L}(n^{-d_*})$ uniformly.

Proof. The positive map $B + \text{Ad}_Q$ acts in dimension $q < d_*$, so its central invariant is nonnegative. Its coefficient expansion is

$$0 \leq \Gamma_q(B + \text{Ad}_Q) = \sum_{k=0}^q k! c_{q-k}(B, Q).$$

The factors for $k = 0, 1$ are one. Consequently

$$\mathcal{W}_B(Q) = \Gamma_q(B + \text{Ad}_Q) - \sum_{k=2}^q (k! - 1) c_{q-k}(B, Q) \geq - \sum_{k=2}^q (k! - 1) |c_{q-k}(B, Q)|.$$

The exterior-power argument, which applies to any linear B with the indicated HS norm, bounds the last coefficient by $(q)_{q-k} \beta^{q-k} e_k(QQ^*) = (q!/k!) \beta^{q-k} e_k(QQ^*)$. This proves (S5.19).

Apply the Gamma transform with $G_n \sim \Gamma(n - q + 1, 1)$ and $Q = \sqrt{G_n}K$. Homogeneity and

$$\frac{\mathbb{E} G_n^k}{(n)_q} = \frac{1}{(n)_{q-k}}$$

give (S5.20). For the final assertion, $e_k(KK^*) \leq \binom{q}{k} L^{2k} n^{-3k/2}$ and $(n)_{q-k}^{-1} = O_q(n^{-q+k})$, uniformly for $n \geq q$. Each summand is $O_{q,L}(n^{-q-k/2})$, whose exponent is at least $q+1 = d_*$ because $k \geq 2$. $\square$

The fourth-order estimate uses only central nonnegativity and the coefficient bounds. It does not assert noncentral nonnegativity. It concerns an entirely soft subcritical core; Proposition S5.13 also permits unrestricted hard singular directions.

S5.4.3. Critical directions near a reduction compression. The noncentral zero set can be controlled for odd reduction compressions. Positivity forces the coupling terms to vanish on that set, and a quantitative version of this cancellation permits small compression errors. Recall the odd reduction map $\mathcal{R}_q$ in (S5.10). The Choi covariance of $\alpha \mathcal{R}_q$ is $\alpha(I - F)/(q - 1)$, where $F(u \otimes v) = v \otimes u$. It is realized by a skew-symmetric Gaussian matrix with independent upper-triangular entries of variance $\alpha/(q - 1)$.

Lemma S5.15 (Coercivity of the odd skew-Gaussian shift). *Let $q \geq 3$ be odd, let Z be a complex skew-symmetric $q \times q$ matrix whose entries Z_{ij} , $i < j$, are independent circular complex Gaussians of variance $\sigma^2 > 0$, and let $Q \in M_q$. Then*

$$(S5.21) \quad \mathbb{E} |\det(Q + Z)|^2 \geq (q - 1)! \sigma^{2(q-1)} \left\| \frac{Q + Q^\top}{2} \right\|_F^2.$$

In particular its zero set is exactly the skew-symmetric matrices.

Proof. Write $Q_{\text{sym}} = (Q + Q^\top)/2$. The terms of distinct degree in Z are orthogonal by circular invariance. The term of degree $q - 1$ is $\text{tr}(\text{adj}(Z)Q)$. For odd skew-symmetric Z , let v_i be its signed principal Pfaffian minors. The cofactor identity gives $\text{adj}(Z) = vv^\top$, and this term equals $v^\top Qv$.

Unitary congruence preserves the distribution of Z . Under $Z \mapsto UZU^\top$, its Pfaffian vector transforms as $v \mapsto \det(U)\bar{U}v$. Its fourth-moment form is consequently a scalar multiple of the Frobenius norm on symmetric matrices:

$$\mathbb{E} |v^\top Qv|^2 = \mathbb{E} |v_1|^4 \|Q_{\text{sym}}\|_F^2.$$

For completeness, independent diagonal phases eliminate covariance between distinct unordered index pairs. A rotation of two coordinates gives $\mathbb{E}|v_1|^4 = 2\mathbb{E}|v_1v_2|^2$, which yields the displayed identity by expanding the symmetric quadratic form.

Writing $q - 1 = 2m$, the elementary Pfaffian moment identity is

$$\mathbb{E}|\text{Pf}(Z_{2m})|^4 = (2m)!\sigma^{4m}.$$

To see this, condition on the first row and rotate its remaining $2m - 1$ coordinates so that just its first entry is nonzero. The remaining skew block is still an independent skew-Gaussian matrix by unitary-congruence invariance. Pfaffian expansion reduces its fourth moment to that in dimension $2m - 2$, multiplied by the second moment of the squared first-row norm. This squared norm has the Gamma distribution with shape $2m - 1$ and scale σ^2 , so the multiplier is $(2m - 1)(2m)\sigma^4$. Starting in dimension zero proves the identity. Therefore $\mathbb{E}|v_1|^4 = (q - 1)!\sigma^{2(q-1)}$. Retaining this nonnegative orthogonal term proves (S5.21). If the expectation is zero, the right side forces Q to be skew-symmetric; conversely $Q + Z$ is then an odd skew-symmetric matrix, so its determinant vanishes. $\square$

Proof of Theorem S5.5. We first prove a pointwise bound for the noncentral polynomial. Write $B = B_n$, $\alpha = \alpha_n$ and $\delta = \delta_n$, suppressing the subscript n . The soft reference covariance is $C_0 = \alpha(I - F)/(q - 1)$. The soft Choi block differs from C_0 in Frobenius norm by at most $q\delta$, since reshuffling identifies Choi Frobenius norm with the superoperator Frobenius norm and the latter is at most q times the HS operator norm. Also $\alpha \leq 2$: the reference map has HS operator norm α , whereas $\|B_{00}\|_{\text{HS} \rightarrow \text{HS}} \leq 1$ and $\delta \leq 1$.

Block positivity controls the covariance between symmetric soft variables and hard-soft or soft-hard variables. Let C be the full Choi matrix, let e_i be a hard basis vector, and let $x \in \mathbb{C}^q$. Block positivity on the product curve $(x + te_i) \otimes x$ gives

$$|\langle e_i \otimes x, C(x \otimes x) \rangle|^2 \leq \langle e_i \otimes x, C(e_i \otimes x) \rangle \langle x \otimes x, C(x \otimes x) \rangle \leq q\delta \|x\|^6.$$

Here the first factor is at most $\|x\|^2$ by the norm bound on B ; the second is at most $q\delta \|x\|^4$ because $C_0(x \otimes x) = 0$. The analogous bound holds for the product curve $x \otimes (x + te_i)$. Comparing the coefficients of the distinct monomials $\overline{x_j}x_kx_l$, or equivalently using norm equivalence on their finite-dimensional polynomial space, shows that every covariance between a symmetric soft variable and a cross-block variable is bounded by $C_r\sqrt{\delta}$. Covariance between a symmetric soft variable and any soft variable is bounded by $C_r\delta$, directly from the soft Choi-block estimate. No small bound on its covariance with a hard-hard variable is asserted.

Independent unitary changes of the hard input and output coordinates reduce a general hard block H to $\text{diag}(t_1, \dots, t_h)$, $t_i \geq 0$, without changing the soft compression or the norm bounds. Put $p_I = \prod_{i \in I} t_i$ for $I \subseteq [h]$. Expansion in these diagonal entries gives

$$\det(H \oplus Q + Z) = \sum_{I \subseteq [h]} p_I f_I(Z, Q),$$

where $f_{[h]} = \det(Q + Z_{00})$. The coefficient $f_{[h] \setminus \{i\}}$ is the determinant on the soft coordinates and hard coordinate i , with hard deterministic entry zero. Thus the top pairing is $a(Q)|p_{[h]}|^2$, where $a(Q) = \mathcal{W}_{B_{00}}(Q)$; its pairings with these h next coefficients have the form $2\text{Re}(p_{[h]}\overline{p_{[h] \setminus \{i\}}}\ell_i(Q))$. They are the hard-linear coefficients for the corresponding $(q + 1)$ -dimensional compressions of B . Each ℓ_i has total degree at most $2q - 1$ in $Q, \overline{Q}$: the covariance-free determinant term contributes only to the top pairing, so a hard-linear term must contain at least one covariance pair.

If Q is skew-symmetric, that pairing obeys

$$(S5.22) \quad |\ell_i(Q)| \leq C_r\sqrt{\delta}(1 + \|Q\|_F)^{2q-1}.$$

Indeed, write the formal matrix on these $q + 1$ coordinates as $\begin{pmatrix} z & r_i \\ c_i & Z_0 \end{pmatrix}$, and put $f = \det(Q + Z_0)$ and $g = zf - r_i \text{adj}(Q + Z_0)c_i$. Expansion in its hard deterministic entry t is $tf + g$. Express Z_0

in symmetric and skew-symmetric coordinates. Every monomial of f contains a symmetric soft variable. In $\Lambda_B(f r_i \operatorname{adj}(\overline{Q + Z_0} c_i))$, this variable must pair with a soft or cross-block variable and therefore supplies a factor $O_r(\sqrt{\delta})$. In $\Lambda_B(f z \overline{f})$, take instead a conjugate symmetric variable from $\overline{f}$. It must pair with an unbarred soft variable in f , supplying a factor $O_r(\delta)$. All other covariance entries are uniformly bounded, and there are finitely many Wick pairings. The polynomial degree is at most $2q - 1$, which proves (S5.22).

Put $Q_{\text{sk}} = (Q - Q^\top)/2$ and $Q_{\text{sym}} = (Q + Q^\top)/2$, and split

$$\ell_i(Q) = \ell_i^{(0)}(Q) + \varepsilon_i(Q), \quad \ell_i^{(0)}(Q) = \ell_i(Q) - \ell_i(Q_{\text{sk}}), \quad \varepsilon_i(Q) = \ell_i(Q_{\text{sk}}).$$

Uniform polynomial derivative bounds and (S5.22) give

$$|\ell_i^{(0)}(Q)| \leq C_r \|Q_{\text{sym}}\|_F (1 + \|Q\|_F)^{2q-2}, \quad |\varepsilon_i(Q)| \leq C_r \sqrt{\delta} (1 + \|Q\|_F)^{2q-1}.$$

Let $a_0(Q)$ denote the noncentral soft polynomial of the reference reduction-transpose map. Lemma S5.15 gives

$$a_0(Q) \geq c_q \alpha_0^{q-1} \|Q_{\text{sym}}\|_F^2, \quad c_q = \frac{(q-1)!}{(q-1)^{q-1}}.$$

The actual top coefficient satisfies

$$|a(Q) - a_0(Q)| \leq C_r \delta (1 + \|Q\|_F)^{2q}.$$

This follows from the polynomial covariance formula and the soft Choi-block norm estimate; its coefficients are Lipschitz on the bounded covariance set under consideration.

Complete the square using $a_0(Q)$ and the terms $\ell_i^{(0)}(Q)$, which is valid also at $a_0(Q) = 0$ because those terms then vanish. The result is bounded below by

$$-C_{r,\alpha_0} (1 + \|Q\|_F)^{4q-4} e_{h-1}(HH^*).$$

The difference between a and a_0 costs at most $C_r \delta (1 + \|Q\|_F)^{2q} |p_{[h]}|^2$. For the remaining top/next terms, Cauchy–Schwarz followed by $2uv \leq u^2 + v^2$ gives

$$2 \left| p_{[h]} \sum_i \overline{p_{[h] \setminus \{i\}}} \varepsilon_i(Q) \right| \leq C_r \delta (1 + \|Q\|_F)^{4q-2} |p_{[h]}|^2 + C_r \sum_i |p_{[h] \setminus \{i\}}|^2.$$

Every other pairing has $|I| + |J| \leq 2h - 2$ and a coefficient polynomial in $Q, \overline{Q}$ of degree at most $2q$, with uniformly bounded coefficients. Its hard monomial satisfies

$$|p_I p_J| \leq e_{h-1}(HH^* + I_h).$$

Put $J_0 = I \cap J$ and split $I \triangle J = D_1 \sqcup D_2$ so that $|J_0 \cup D_\nu| \leq h - 1$ for both ν ; the split exists because $2|J_0| + |I \triangle J| \leq 2h - 2$. The arithmetic–geometric mean inequality gives

$$|p_I p_J| \leq \frac{1}{2} \sum_{\nu=1}^2 \prod_{i \in J_0 \cup D_\nu} t_i^2.$$

Each product occurs with a positive integer coefficient in $e_{h-1}(HH^* + I_h)$. This proves the bound for every remaining pairing. Combining these estimates proves the pointwise bound

$$(S5.23) \quad \mathcal{W}_B(H \oplus Q) \geq -C_{r,\alpha_0} (1 + \|Q\|_F)^{4q-2} (\delta \det(HH^*) + e_{h-1}(HH^* + I_h)).$$

In particular the square completion never assumes that the actual coefficient $a(Q)$ is nonnegative.

In the Gamma transform set $H = \sqrt{G_n} H_n$ and $Q = \sqrt{G_n} R_n$. Admissibility and the soft-norm bound imply

$$\det(HH^*) \leq G_n^h, \quad e_{h-1}(HH^* + I_h) \leq h(1 + G_n)^{h-1}, \quad \|Q\|_F \leq L \sqrt{G_n/n}.$$

For each fixed nonnegative integer m ,

$$\mathbb{E}(G_n/n)^m = \prod_{j=0}^{m-1} \frac{n-r+1+j}{n} = O_{r,m}(1).$$

These moment bounds give the upper bound $C_{r,\alpha_0,L}(\delta n^h + n^{h-1})/(n)_r$ on the negative part. For $n \geq r$, the ratios $n^h/(n-q)_h$ and $n^{h-1}/(n-q-1)_{h-1}$ are bounded in terms of r . This yields (S5.11). Finally, $\delta \leq M/n$ implies $\delta/(n)_q \leq M/(n)_{q+1}$, proving the stated consequence. $\square$

The reduction strength must remain bounded away from zero. This does not prove the bound for an arbitrary positive soft compression.

S6. CONTRACTION, CONCENTRATION, AND THE STRUCTURE OF MINIMIZERS

The bounds of the paper constrain the orbital weights. This section constrains the map itself: its contraction radius on traceless matrices bounds $|\Phi_n|$ and proves $\mu_2 = 1/4$; the concentration estimate (S5.12) shows that negativity of order $1/n$ needs a map close to an isometry; and capacity normalization identifies the first-order conditions at any negative global minimizer. The decomposable zero bound and the examples for the entanglement-breaking criterion are included here because they share these tools.

S6.1. Bounds from determinant energy and contraction. We use the Fischer product and determinant pairing from (A.3).

S6.1.1. An adjoint-energy lower bound. For an arbitrary complex-linear map L , let $\mathcal{U}_L f = f \circ L$ on homogeneous polynomials. The symmetric-tensor realization of the Fischer product gives

$$\mathcal{U}_L^* = \mathcal{U}_{L^*}, \quad \mathcal{U}_A \mathcal{U}_B = \mathcal{U}_{B \circ A}.$$

Thus the energy

$$\mathcal{E}(H) := \Phi_n(HH^*) = \frac{1}{n!} \|p \circ H\|_F^2 = \frac{1}{n!} \mathbb{E} |\det H(G)|^2$$

measures the norm of the determinant pullback. Products of maps in this section mean composition, and H^* is the Hilbert–Schmidt adjoint.

Proposition S6.1. *Let $H : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ be positive and bistochastic, and put $b = n^{-n}$. Then*

$$b \leq \mathcal{E}(H) \leq 1, \quad \Phi_n(H) \geq b - \sqrt{(1-b)(\mathcal{E}(H) - b)}.$$

The second bound also holds with $\Phi_n(H^*H)$ in place of $\mathcal{E}(H)$.

Proof. Lemma A.1 makes H and its symmetric tensor powers contractive, so $\|p \circ H\|_F^2 \leq n!$.

Write $X = zI/\sqrt{n} + X_0$, where $\text{tr } X_0 = 0$. Bistochasticity preserves this orthogonal decomposition. The common pure-scalar term of p and $p \circ H$ is $n^{-n/2} z^n$, whose squared Fischer norm is $n!b$. All remaining monomials are orthogonal to it. Subtracting this term from both polynomials and applying Cauchy–Schwarz gives

$$|\Phi_n(H) - b|^2 \leq (1-b)(\mathcal{E}(H) - b).$$

The retained term also proves $\mathcal{E}(H) \geq b$. The same argument applies to H^* , since $\Phi_n(H^*) = \Phi_n(H)$. $\square$

S6.1.2. *Retaining the contraction on traceless matrices.* The scalar term alone loses the positive contribution of degree two. To retain it, let $\Delta_n(X) = \text{tr}(X)I/n$ and define

$$\rho(H) = \|H - \Delta_n\|_{\text{HS} \rightarrow \text{HS}}, \quad q = q(H), \quad F_n(r) = \sum_{k=0}^n r^k \binom{1-r}{n}^{n-k} \quad (0 \leq r \leq 1).$$

Here $\|\cdot\|_{\text{HS} \rightarrow \text{HS}}$ is the operator norm on the matrix Hilbert space, and $q(H)$ is the squared Hilbert–Schmidt norm of the superoperator defined in Section 6. The equality with $\|C_H\|_F^2$ merely rearranges the entries of the matrix of H .

Theorem S6.2. *Let $H : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ be positive and bistochastic, with $n \geq 2$. Put $\rho = \rho(H)$ and $q = q(H)$. Then $0 \leq \rho \leq 1$ and*

$$(S6.1) \quad |\Phi_n(H)| \leq F_n(\rho),$$

$$(S6.2) \quad \Phi_n(H) \geq \frac{2}{n^n} + \frac{n+1}{2n^{n-1}}\rho^2 + \frac{q-1}{2(n-1)n^{n-1}} - F_n(\rho).$$

Any upper bound for ρ in $[0, 1]$ may replace ρ in both formulas. Equality holds in (S6.2) when $n = 2$.

Proof. Lemma A.1 and bistochasticity give $H = 1 \oplus T$ on the orthogonal decomposition $M_n = \mathbb{C}I \oplus M_n^0$, where M_n^0 is the traceless subspace. Consequently,

$$\rho = \|T\| \leq 1, \quad \|T\|_F^2 = q - 1.$$

The second identity uses invariance of the Frobenius norm under Choi reshuffling. Expand the determinant by degree in X_0 :

$$p(zI/\sqrt{n} + X_0) = \sum_{k=0}^n z^{n-k} p_k(X_0), \quad a_k = \frac{(n-k)!}{n!} \|p_k\|_F^2.$$

Fischer orthogonality and the norm of the degree- k tensor power imply

$$(S6.3) \quad \Phi_n(H) = \sum_{k=0}^n \frac{(n-k)!}{n!} \langle p_k, p_k \circ T \rangle_F, \quad |\Phi_n(H)| \leq \sum_{k=0}^n a_k \rho^k.$$

For $T = r \text{Id}$, every term in the last sum is attained. The identity

$$\Phi_n(a \text{Id} + b \text{tr}(\cdot)I) = \sum_{k=0}^n a^k b^{n-k}$$

therefore identifies $\sum_k a_k r^k$ with $F_n(r)$. To verify the identity, expand the output determinant as $\sum_k a^k b^{n-k} e_k(X) (\text{tr } X)^{n-k}$. The normalized determinant differential extracts the column-Schur coefficient, which is one in each $e_k(X) (\text{tr } X)^{n-k}$. Indeed, multiplication of a Schur polynomial by $\text{tr } X = s_{(1)}(X)$ adds one cell in each allowed position, with coefficient one; starting from $e_k = s_{(1^k)}$, there is exactly one path to the column (1^n) . This is the one-box Pieri rule (Macdonald, 1995, Chapter I, §5). This proves (S6.1).

The first three terms can be computed before applying the norm bound:

$$a_0 = n^{-n}, \quad p_1 = 0, \quad p_2(X_0) = -\frac{\text{tr}(X_0^2)}{2n^{(n-2)/2}}, \quad a_2 = \frac{n+1}{2n^{n-1}}.$$

In a Hermitian orthonormal basis of M_n^0 , the matrix of T is real and $\text{tr}(X_0^2) = \sum_j z_j^2$. Gaussian monomial orthogonality then gives

$$\frac{(n-2)!}{n!} \langle p_2, p_2 \circ T \rangle_F = \frac{\|T\|_F^2}{2(n-1)n^{n-1}}.$$

Bounding the terms of degree at least three in (S6.3) from below by $-\sum_{k=3}^n a_k \rho^k$ proves (S6.2). This sum is empty for $n = 2$. Since $a_k \geq 0$, increasing ρ increases both $\sum_k a_k \rho^k$ and $\sum_{k=3}^n a_k \rho^k$, proving the assertion about an upper bound for ρ .

For $n = 2$, the equality reduces to $\Phi_2(H) = q(H)/4$. The scalar direction forces $q(H) \geq 1$, and Δ_2 attains equality, so $\mu_2 = 1/4$. $\square$

The polynomial F_n distinguishes a fixed contraction gap from the unrestricted problem. If $\rho(H_n) \leq \rho_0 < 1$, with $\rho_0 > 0$ fixed, then its geometric-sum expression gives

$$|\Phi_n(H_n)| \leq \rho_0^n (1 + O_{\rho_0}(n^{-1})).$$

Conversely, if $\Phi_n(H_n) \leq -cn^{-p}$ for fixed $c, p > 0$, then

$$(S6.4) \quad 1 - \rho(H_n) \leq \frac{p \log n - \log c + O(n^{-1})}{n}.$$

Indeed, the first bound forces $\rho(H_n) > 1/2$ eventually. For $\rho > 1/2$, one has $F_n(\rho) \leq \rho^n/(1 - 1/n)$; taking logarithms and using $\log \rho \leq -(1 - \rho)$ proves (S6.4).

S6.1.3. The decomposable zero bound. An additional algebraic hypothesis can remove the negative part entirely. A map is decomposable if it admits a representation

$$H(X) = \sum_a A_a X A_a^* + \sum_b B_b X^T B_b^*.$$

The mixed Gaussian formula of Sra (2026) gives the bound without a bistochastic assumption.

Proposition S6.3. *Suppose H has the displayed decomposable representation. For independent standard circular complex Gaussians g_a, h_b , set $Z = \sum_a g_a A_a$ and $W = \sum_b h_b B_b$. Then*

$$n! \Phi_n(H) = \mathbb{E} \det \begin{pmatrix} Z & -W \\ \overline{W} & \overline{Z} \end{pmatrix} \geq 0.$$

Proof. In the determinant expansion, Gaussian contraction pairs each top row α with a bottom row $\tau(\alpha)$. The paired columns must lie in opposite blocks. Summing over the two orientations gives

$$\mathbb{E} \det \begin{pmatrix} Z & -W \\ \overline{W} & \overline{Z} \end{pmatrix} = \sum_{\tau, i, j \in \mathfrak{S}_n} \text{sgn}(\tau) \text{sgn}(i) \text{sgn}(j) \prod_{\alpha=1}^n H(E_{i(\alpha), j(\alpha)})_{\alpha, \tau(\alpha)}.$$

The last expression is the Leibniz expansion of $n! \Phi_n(H)$. The block matrix commutes with the antiunitary $(u, v) \mapsto (-\bar{v}, \bar{u})$, whose square is $-I$. Nonreal eigenvalues occur in conjugate pairs, and the generalized eigenspace of each real eigenvalue has even complex dimension. Its determinant is therefore nonnegative, including when the matrix is singular. $\square$

Complete positivity and complete copositivity are special cases of this proposition. Ordinary positivity alone does not supply either Kraus representation.

S6.2. Consequences of the entanglement-breaking bound. The tail estimate (S5.7) has two asymptotic regimes. For fixed $\eta > 0$, it is $\eta^n e^{(1-\eta)/\eta} (1 + O_\eta(n^{-1}))/n$. For fixed $n \geq d_*$ and $\eta \downarrow 0$, it is $\eta^{d_*}/[d_*(n - d_*)!] + O_n(\eta^{d_*+1})$.

The entanglement-breaking criterion permits negative maps with $\rho(H) = 1$. Choose $B_r \in \mathfrak{B}_r$ with $\Phi_r(B_r) < 0$, let D_k denote coordinate dephasing on M_k , and fix $\eta < 1$ sufficiently close to one that $A_r = \eta B_r + (1 - \eta) D_r$ still has $\Phi_r(A_r) < 0$. For $n > r$, put

$$H_n = A_r \boxplus (\eta \text{Id}_{n-r} + (1 - \eta) D_{n-r}).$$

The entanglement-breaking submap $(1 - \eta) D_n$ gives $b_{\text{EB}}(H_n) \geq 1 - \eta$. The map fixes a nonzero traceless matrix scalar on each block, so $\rho(H_n) = 1$. With $m = n - r$, the direct-sum identity and the expansion (S5.4) give

$$\Phi_n(H_n) = \frac{\Phi_r(A_r)}{\binom{n}{r}} \eta^m \sum_{\ell=0}^m \frac{((1 - \eta)/\eta)^\ell}{\ell!} < 0.$$

Thus the entanglement-breaking criterion gives an exponential bound on a negative family to which the strict-contraction estimate does not apply.

Conversely, if $\Phi_n(H_n) \leq -cn^{-p}$ for fixed $c > 0$ and $p \geq 1$, then (S5.5) implies, for all sufficiently large n ,

$$b_{\text{EB}}(H_n) \leq \frac{(p-1) \log n + \log(e(1+1/d_*)/c)}{n}.$$

Indeed the alternative $b_{\text{EB}} \geq 1/2$ would make the upper bound on the negative part exponentially small, and the remaining case follows by taking logarithms. Thus negativity of order $1/n$ requires $b_{\text{EB}} = O(1/n)$.

S6.3. Concentration across input bases. The orbital weights have a different probabilistic interpretation. For an input basis $U = (u_i)$, put $A_i = H(u_i u_i^*)$. Decompose these matrices into rank-one terms and group the resulting columns by colors indexed by i . Their concatenation is a matrix V with $VV^* = I$. For each set S of n columns, Cauchy–Binet assigns probability $|\det V_S|^2$, and its generating polynomial is $\det(\sum_i x_i A_i)$. If M is the number of colors absent from S , its monomial has support size $n - M$. Equation (2.3), averaged over both S and U , therefore gives

$$(S6.5) \quad \Phi_n(H) = \mathbb{E} \frac{(-1)^M}{\binom{n}{M}}.$$

Put

$$q = \|H\|_{\text{HS}, \text{super}}^2, \quad d = \frac{n^2 - q}{n + 1}.$$

Lemma A.1 and the fixed scalar direction imply $1 \leq q \leq n^2$ and $0 \leq d \leq n - 1$. The parameter d is the mean total impurity of the images of an orthonormal basis of rank-one projections.

The same parameter also gives an exact linear constraint on the orbital weights:

$$(S6.6) \quad \sum_{\lambda \vdash n} \left(\sum_i \lambda_i (\lambda_i - 1) \right) w_\lambda = d.$$

Indeed, for a Haar unit vector u , the identity $\mathbb{E}(u_i \bar{u}_j u_k \bar{u}_l) = (\delta_{ij} \delta_{kl} + \delta_{il} \delta_{kj}) / (n(n+1))$ gives $\mathbb{E}_u \text{tr } H(uu^*)^2 = (n+q)/(n(n+1))$; here δ_{ij} is one if $i = j$ and zero otherwise. Comparing the coefficient of t^{n-2} in $f_H(t1^n + 1_{[1]})$ gives

$$\frac{1}{n} \sum_{\lambda} \sum_i \binom{\lambda_i}{2} w_\lambda = \mathbb{E}_u e_2(H(uu^*)) = \frac{1}{2} \left(1 - \frac{n+q}{n(n+1)} \right),$$

which is exactly (S6.6).

The statistic on the left of (S6.6) vanishes at (1^n) and is at least two elsewhere. Thus $w_{(1^n)} \geq 1 - d/2$, and Proposition 2.1 implies the additional bound

$$\Phi_n(H) \geq \frac{q - n^2 + 2n}{2n}.$$

In particular, $q \geq n^2 - 2n$ ensures nonnegativity. The exponential estimate below gives different information when d grows with n .

Theorem S6.4. For $H \in \mathfrak{B}_n$ and $n \geq 2$, with $d = (n^2 - q(H))/(n+1)$,

$$(S6.7) \quad \Phi_n(H) \geq -\frac{3}{n} e^{-d/64}.$$

For $n \geq 6$, one also has

$$(S6.8) \quad \Phi_n(H) \geq -\frac{2}{n} e^{-d/(6+8/n)} - \binom{n}{3}^{-1} - n^{-n} - (2/n)^n.$$

Proof. For an input basis $U = (u_i)$, write $P_i = u_i u_i^*$ and $A_i = H(P_i)$. The rank-one decomposition used for (S6.5) gives an $n \times N$ matrix V with $VV^* = I$; its columns are grouped by color i . The measure assigning mass $|\det V_S|^2$ to each n -element set S is a projection determinantal measure. Such measures are negatively associated (Borcea et al., 2009, Proposition 3.5 and Theorem 4.9): for nonnegative increasing functions depending on disjoint groups of occupancy indicators, the expectation of their product is at most the product of their expectations.

Let M count missing colors. The probability that color i is missing is $p_i = \det(I - A_i)$. For $0 < t < 1$, the functions $t^{\mathbf{1}_{\{i \text{ missing}\}}}$ are nonnegative increasing functions of the microscopic occupancies in disjoint color classes. Negative association therefore gives

$$(S6.9) \quad \mathbb{E}[t^M \mid U] \leq \prod_i (1 - (1-t)p_i) \leq \exp\left(- (1-t) \sum_i p_i\right).$$

For a trace-one matrix $A \succeq 0$, the elementary symmetric functions of its eigenvalues satisfy

$$\det(I - A) \geq e_2(A) - e_3(A) \geq \frac{2}{3}e_2(A) = \frac{1}{3}(1 - \operatorname{tr} A^2).$$

The first inequality is Bonferroni's inequality applied to independent Bernoulli variables with those eigenvalue parameters; the second follows from $e_1 e_2 \geq 3e_3$ and $e_1 = 1$. Thus, with

$$d_U = n - \sum_i \operatorname{tr} A_i^2,$$

equation (S6.9) at $t = 1/2$ yields, for $a \geq 0$,

$$(S6.10) \quad \Pr(M \leq a \mid U) \leq 2^a e^{-d_U/6}.$$

To average this estimate, view H as a contraction on the real Hilbert space of Hermitian matrices and put $B = \operatorname{Id} - H^*H$, with Id the identity on that Hilbert space. Then $0 \preceq B \preceq I$ and $d_U = \sum_i \langle P_i, B P_i \rangle$. Along the curve $U(t) = e^{tZ}U$ with $Z^* = -Z$, one has $P'_i(0) = [Z, P_i]$. Differentiation therefore gives

$$|\partial_Z d_U| \leq 2\sqrt{d_U} \left(\sum_i \|B^{1/2}[Z, P_i]\|_{\text{HS}}^2 \right)^{1/2} \leq 2\sqrt{2d_U} \|Z\|_{\text{HS}}.$$

The last inequality follows in the basis U by summing the off-diagonal entries of Z . Hence $\|\nabla d_U\|^2 \leq 8d_U$. The pure-state second moment gives

$$(S6.11) \quad \mathbb{E} d_U = n - \frac{n+q}{n+1} = d.$$

The function d_U is invariant under scalar phases, so its distribution under Haar $U(n)$ equals its distribution under Haar $SU(n)$. Write $\operatorname{Ent}(Y) = \mathbb{E}(Y \log Y) - \mathbb{E}Y \log \mathbb{E}Y$ for the entropy of a nonnegative random variable. For the unnormalized Hilbert–Schmidt metric, $SU(n)$ satisfies

$$\operatorname{Ent}(g^2) \leq \frac{4}{n} \mathbb{E} \|\nabla g\|^2;$$

this is the $SU(n)$ constant in the proof of Meckes and Meckes's Theorem 15 (Meckes and Meckes, 2013). For $s > 0$, put $\psi(s) = \log \mathbb{E} e^{-s d_U}$. Applying this inequality to $g = e^{-s d_U/2}$ gives

$$s\psi'(s) - \psi(s) \leq -\frac{8s^2}{n} \psi'(s).$$

Therefore $((1 + 8s/n)\psi(s)/s)' \leq 0$. Its limit at zero is $-d$, by (S6.11), and consequently

$$(S6.12) \quad \mathbb{E} e^{-s d_U} \leq \exp\left(-\frac{s d}{1 + 8s/n}\right).$$

Combining (S6.10) and (S6.12) gives $\Pr(M \leq a) \leq 2^a e^{-d/(6+8/n)}$.

The opposite end of the occupancy distribution is controlled by projection images. Let W_ℓ be the probability of exactly ℓ occupied colors. For a rank- ℓ projection P , trace preservation and arithmetic–geometric mean give $\det H(P) \leq (\ell/n)^n$. Evaluating the orbital polynomial at $(1^\ell, 0^{n-\ell})$ and discarding nonnegative terms yields

$$\frac{W_\ell}{\binom{n}{\ell}} \leq (\ell/n)^n.$$

Split the signed expectation (S6.5) into $M < r$, $r \leq M \leq n - r$, and $n - M < r$. For integers $1 \leq r \leq \lfloor n/2 \rfloor$, this gives

$$(S6.13) \quad \Phi_n(H) \geq -\frac{2^{r-1}}{n} e^{-d/(6+8/n)} - \binom{n}{r}^{-1} - \sum_{\ell=1}^{r-1} (\ell/n)^n.$$

When $r = 3$, only $M = 1$ contributes negatively in the first interval, so its factor is 2, proving (S6.8).

If $d < 16$, the elementary bound $\Phi_n \geq -1/n$ implies (S6.7). If $d \geq 16$, then $n \geq 17$; choose $r = \lfloor d/16 \rfloor + 1$. The three negative terms in (S6.13) are each at most $e^{-d/64}/n$ in magnitude. Indeed,

$$\begin{aligned} 2^{r-1} e^{-d/(6+8/n)} &\leq e^{(\log 2/16 - 2/13)d} \leq e^{-d/64}, \\ \frac{n}{\binom{n}{r}} = \frac{r}{\binom{n-1}{r-1}} &\leq \frac{r}{16^{r-1}} \leq 8^{-(r-1)} \leq e^{-d/64}, \\ n \sum_{\ell=1}^{r-1} (\ell/n)^n &\leq n^2 16^{-n} \leq 8^{-n} \leq e^{-d/64}. \end{aligned}$$

Here $r - 1 \leq (n - 1)/16$, $r \leq 2^{r-1}$, and $d < 16r$ justify the middle line. Summing proves (S6.7). $\square$

For fixed $c > 0$ and $p \geq 1$, the condition $\Phi_n(H_n) \leq -cn^{-p}$ implies

$$d(H_n) \leq 64((p - 1) \log n + \log(3/c)).$$

Thus polynomial negativity requires $n^2 - q = O(n \log n)$, and order- $1/n$ negativity requires $n^2 - q = O(n)$. These restrictions leave the padded examples available: for a fixed $H_r \in \mathfrak{B}_r$ and its pinched direct sum $H_r \boxplus \text{Id}_{n-r}$, the squared superoperator norm is $q_r + (n - r)^2$, where $q_r = \|H_r\|_{\text{HS}, \text{super}}^2$, so $d \rightarrow 2r$.

Bounded d does not force a positive constant lower bound on the square-free mass. For rank-one P , let $\Pi(X) = PXP + (I - P)X(I - P)$. Then

$$q(\Pi) = 1 + (n - 1)^2, \quad d(\Pi) = \frac{2(n - 1)}{n + 1}, \quad w_{(1^n)} = \frac{2}{n + 1}.$$

To compute the last quantity, write $\alpha_i = \|Pu_i\|^2$. Complementary unitary minors give square-free mass $\sum_i \alpha_i^2$ in a fixed basis, and Haar averaging gives $\mathbb{E} \alpha_i^2 = 2/(n(n + 1))$. Only the shapes (1^n) and $(2, 1^{n-2})$ occur, and their signed score is $\Phi_n(\Pi) = 1/n > 0$. The concentration estimate therefore leaves a distinction between small square-free mass and negative invariant.

S6.4. Capacity and the structure of minimizers. The lower certificates also constrain maps that attain the optimal value. Capacity first removes the marginal constraints; the resulting normalization then identifies the first variation at every negative global minimizer.

The elementary floor $L_n = (n + 1)!/n^{n+1} - 1/n$ of (1.2) is never attained for $n \geq 3$:

$$(S6.14) \quad \mu_n > L_n.$$

If equality held, the proof of Proposition 1.1 would force $w_{(1^n)} = n!/n^n$. The equality case of Gurvits's theorem (Gurvits, 2006, Theorem 4.1) would then give $H(u_i u_i^*) = I/n$ for almost every basis. Continuity extends this to every basis, so $H = \Delta_n$. But $\Phi_n(\Delta_n) = n^{-n} > L_n$ for $n \geq 3$, a

contradiction. Compactness turns nonattainment into the strict inequality (S6.14); this argument gives no effective gap.

S6.4.1. *Capacity normalization.* The capacity in (A.2) permits a more precise version of Lemma 2.2: its determinant factor gives an upper bound and the equality conditions at negative minima.

Proposition S6.5. *Every ordinary positive map $K : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ satisfies*

$$(S6.15) \quad \mu_n \text{cap}(K) \leq \Phi_n(K) \leq \text{cap}(K).$$

If $\ell_n \leq 0$ is a universal lower bound on $\mathfrak{B}_n$, then

$$(S6.16) \quad \Phi_n(K) \geq \ell_n \text{cap}(K) \geq \ell_n \det K(I) \geq \ell_n \left(\frac{\text{tr } K(I)}{n} \right)^n.$$

In particular, $\text{cap}(K) = 0$ implies $\Phi_n(K) = 0$.

Proof. Call K strictly positivity improving if $K(X) \succ 0$ for every nonzero $X \succeq 0$. Compactness of the trace-one positive semidefinite matrices makes this equivalent to $K(X) \succeq c \text{tr}(X)I$ for some $c > 0$. For such a K , the minimizing input and normalized map constructed in Lemma 2.2 give

$$\Phi_n(K) = \text{cap}(K) \Phi_n(\widehat{K}), \quad \widehat{K} \in \mathfrak{B}_n.$$

The Fischer contraction bound in Section S6.1 gives $|\Phi_n(\widehat{K})| \leq 1$, proving (S6.15) in this case.

For general K , apply the preceding argument to $K_\varepsilon = K + \varepsilon \text{tr}(\cdot)I$. Monotonicity gives $\text{cap}(K_\varepsilon) \geq \text{cap}(K)$; testing any fixed determinant-one input gives the matching upper limit as $\varepsilon \downarrow 0$. Polynomial continuity of Φ_n proves (S6.15) and the zero-capacity assertion. Finally,

$$\text{cap}(K) \leq \det K(I) \leq \left(\frac{\text{tr } K(I)}{n} \right)^n,$$

so multiplication by $\ell_n \leq 0$ proves (S6.16). $\square$

The capacity of a bistochastic map is one. To see this, diagonalize $X = \sum_i x_i u_i u_i^*$ and $H(X)$ in a frame (v_j) , and put $a_{ji} = v_j^* H(u_i u_i^*) v_j$. The matrix a is doubly stochastic, so weighted arithmetic–geometric mean gives

$$\det H(X) = \prod_j \sum_i a_{ji} x_i \geq \prod_{j,i} x_i^{a_{ji}} = \det X.$$

The input $X = I$ gives equality.

S6.4.2. *The trace normalization and the boundary.* A Hermitian matrix C is block positive if $z^* C z \geq 0$ for every product vector $z = \bar{x} \otimes y$. Positive maps correspond to block-positive Choi matrices, and we write $\Phi_n(C) = \Phi_n(H)$ when $C = C_H$. Consider the trace-normalized set

$$\mathfrak{T}_n = \{C = C^* : C \text{ is block positive, } \text{tr } C = n\}.$$

Proposition S6.6. *For $n \geq 3$,*

$$\min_{C \in \mathfrak{T}_n} \Phi_n(C) = \mu_n.$$

If $\mu_n < 0$, every minimizer on $\mathfrak{T}_n$ is the Choi matrix of a bistochastic map.

Proof. The rank-three reduction-transpose map $R_3(X) = (\text{tr}(X)I - X^\top)/2$ satisfies

$$\det R_3(X) = \frac{\text{tr}(X)e_2(X) - \det X}{8}, \quad \Phi_3(R_3) = 0.$$

The second identity follows because both terms in the numerator have determinant differential equal to $3!$. Its pinched direct sum with an identity map has zero invariant in every dimension $n \geq 3$, so $\mu_n \leq 0$. For $C_K \in \mathfrak{T}_n$, Proposition S6.5 therefore gives $\Phi_n(K) \geq \mu_n$. The reverse inequality follows from the inclusion of the bistochastic Choi matrices in $\mathfrak{T}_n$.

Suppose $\mu_n < 0$ and $\Phi_n(K) = \mu_n$. Since $\text{tr } K(I) = n$, equality in the chain

$$\mu_n = \Phi_n(K) \geq \mu_n \text{cap}(K) \geq \mu_n \det K(I) \geq \mu_n$$

forces $\text{cap}(K) = \det K(I) = 1$. Equality in the trace–determinant inequality gives $K(I) = I$. The input $X = I$ now minimizes the capacity objective, and differentiation gives $K^*(I) = I$. $\square$

Rank-one pinching permits a separate test of a minimizer's position in the positive cone. For a rank-one projection P , write $Q = I - P$ and define

$$\mathcal{D}_{P,t}(X) = PXP + QXQ + t(PXQ + QXP).$$

Each map $\mathcal{D}_{P,t}$ is bistochastic and completely positive for $|t| \leq 1$: it is the convex combination $((1+t)/2)\text{Id} + ((1-t)/2)\text{Ad}_{P-Q}$. An input cut is nonconstant for H if $H \circ \mathcal{D}_{P,0} \neq H$; for an output cut the condition is $\mathcal{D}_{P,0} \circ H \neq H$. A parameter is admissible when the corresponding composition is positive.

Proposition S6.7. *Every global minimizer in $\mathfrak{B}_n$ lies on the boundary of the ordinary-positive cone when $n \geq 3$. Choose such a minimizer H with least $\|C_H\|_{\text{HS}}$. For every nonconstant rank-one input or output cut, write $\Phi_n(H \circ \mathcal{D}_{P,t}) = a_P + b_P t^2$, with the analogous formula for an output cut. Then $b_P < 0$, and the largest admissible nonnegative parameter is $t = 1$.*

Proof. In the determinant contraction, a permutation crosses a rank-one cut either zero or one times in each direction. The dependence on t is therefore $a_P + b_P t^2$. If H were strictly positivity improving, openness would make parameters on both sides of $t = 1$ feasible. Minimality would then force $b_P = 0$. Pinching at $t = 0$ would preserve both the minimum and strict positivity. Repeating the argument for coordinate cuts on the input and output gives a classical bistochastic channel whose invariant is $\text{per}(A)/n!$ for a strictly positive doubly stochastic matrix A , where $\text{per}(A) = \sum_{\sigma \in \mathfrak{S}_n} \prod_i A_{i,\sigma(i)}$. This contradicts $\mu_n \leq 0$.

By Lemma A.1, a minimizer with least Choi norm exists. For the selected minimizer, $b_P > 0$ would decrease the objective under pinching. If $b_P = 0$ on a nonconstant cut, the identity

$$\|C_{H \circ \mathcal{D}_{P,t}}\|_{\text{HS}}^2 = \|C_{H \circ \mathcal{D}_{P,0}}\|_{\text{HS}}^2 + t^2 \|C_{H - H \circ \mathcal{D}_{P,0}}\|_{\text{HS}}^2$$

would contradict the choice of Choi norm. Thus $b_P < 0$, and any feasible $t > 1$ would decrease the objective. The output argument is identical. $\square$

S6.4.3. A separable cofactor at negative minima. The trace normalization makes the Lagrange multiplier scalar. Let $\text{Sep}_{n,n}$ be the cone generated by positive product projectors, and define the Hermitian cofactor $G(C)$ by

$$D\Phi_n(C)[D] := \left. \frac{d}{dt} \Phi_n(C + tD) \right|_{t=0} = \text{tr}(G(C)D) \quad (D = D^*).$$

The separable cone is the dual of the block-positive cone under the trace pairing.

Theorem S6.8. *Suppose $\mu_n < 0$ and let $C_\star$ be the Choi matrix of any global minimizer in $\mathfrak{B}_n$. Then*

$$G(C_\star) = \mu_n I_{n^2} + Z_\star, \quad Z_\star \in \text{Sep}_{n,n}, \quad \text{tr}(Z_\star C_\star) = 0.$$

The slack admits a decomposition

$$Z_\star = \sum_{j=1}^N t_j z_j z_j^*, \quad N \leq n^4, \quad t_j \geq 0, \quad z_j = \bar{x}_j \otimes y_j,$$

in which $z_j^* C_\star z_j = 0$ whenever $t_j > 0$.

Proof. By Proposition S6.6, $C_\star$ minimizes Φ_n on $\mathfrak{T}_n$. Its first variation consequently satisfies

$$\mathrm{tr}(G(C_\star)(D - C_\star)) \geq 0 \quad (D \in \mathfrak{T}_n).$$

Euler's identity gives $\mathrm{tr}(G(C_\star)C_\star) = n\mu_n$. Every nonzero block-positive matrix B has $\mathrm{tr} B > 0$: its nonnegative product-vector form has positive integral unless it vanishes identically, in which case $B = 0$ because product projectors span the Hermitian matrices. Substituting $D = nB/\mathrm{tr} B$ in the preceding inequality gives

$$\mathrm{tr}((G(C_\star) - \mu_n I)B) \geq 0.$$

Duality of the two cones therefore gives $Z_\star = G(C_\star) - \mu_n I \in \mathrm{Sep}_{n,n}$. Euler's identity also gives $\mathrm{tr}(Z_\star C_\star) = 0$.

Conic Carathéodory gives a decomposition into at most n^4 product projectors. Every summand pairs nonnegatively with $C_\star$, so their zero sum forces the stated product contacts. $\square$

The cofactor equation can certify a proposed floor by excluding all lower levels. For this purpose its separable slack must be retained; positive semidefiniteness alone gives a weaker necessary condition.

Corollary S6.9. *For $c \leq 0$, the inequality $\Phi_n(H) \geq c$ for every $H \in \mathfrak{B}_n$ is equivalent to nonexistence of a triple (C, λ, Z) satisfying*

$$C = C_H \quad (H \in \mathfrak{B}_n), \quad \lambda < c, \quad Z \in \mathrm{Sep}_{n,n}, \quad G(C) = \lambda I + Z, \quad \mathrm{tr}(ZC) = 0.$$

Proof. Euler's identity gives $\Phi_n(C) = \lambda$ for every such triple. Conversely, a violation of the proposed floor gives $\mu_n < c \leq 0$, and Theorem S6.8 supplies a triple at a global minimizer. $\square$

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