

A SPECTRAL BLAKLEY–ROY INEQUALITY FOR NONNEGATIVE MATRICES

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ABSTRACT. Let A and B be real symmetric matrices with nonnegative entries. We prove that $\rho(AB)^k \leq \rho(A^k B^k)$ for the spectral radius $\rho(\cdot)$ and every integer $k \geq 1$; for $B = uu^T$ this is the Blakley–Roy inequality. We prove it via the more general inequality $\|(AB)^m\|^k \leq \|(A^k B^k)^m\|$ in the spectral norm, valid for all integers $k, m \geq 1$. The even exponents case follows from standard tools; for odd exponents $k = 2r + 1$ the claim follows via the more general product inequality, $(u^T C_1 \cdots C_t v)^k \leq (u^T u)^r (v^T v)^r u^T C_1^k \cdots C_t^k v$ for nonnegative symmetric C_i and nonnegative u, v . Its proof is an entropy argument on paths that traverse each edge k times, back and forth.

1. INTRODUCTION

For positive semidefinite A and B , the powers $(AB)^k$ and $A^k B^k$ satisfy numerous spectral inequalities. Two classical examples are: (i) the Cordes inequality $\|A^t B^t\| \leq \|AB\|^t$ for $0 \leq t \leq 1$ (Cordes, 1987; Furuta, 1989) which implies the operator norm inequality $\|AB\|^k \leq \|A^k B^k\|$; (ii) and the Araki–Lieb–Thirring inequality $\text{tr}(B^{1/2} AB^{1/2})^q \leq \text{tr}(B^{k/2} A^k B^{k/2})^q$ for $q > 0$ (Araki, 1990), which in the limiting case as $q \rightarrow \infty$ gives the spectral radius inequality $\rho(AB)^k \leq \rho(A^k B^k)$.

But what about such inequalities without assuming semidefiniteness? Pourya Habibzadeh conjectured (personal communication, ca. 2019) that $\rho(AB)^k \leq \rho(A^k B^k)$ remains true when A and B are symmetric with nonnegative entries. When B is a rank-one semidefinite matrix $B = uu^T$ with $u \geq 0$, we have $\rho(AB)^k = (u^T A u)^k$ and $\rho(A^k B^k) = (u^T u)^{k-1} u^T A^k u$. So the conjecture contains the Blakley–Roy inequality (Blakley and Roy, 1965) as a special case. The Blakley–Roy inequality is also in Mulholland and Smith (1959) and London (1966), as Erdős and Simonovits (1982) note, and it is the path case of Sidorenko’s conjecture (Sidorenko, 1993).

If instead of the spectral radius $\rho(\cdot)$, we ask the same question for operator norms, then already Cordes’ inequality supplies a proof. Indeed, with $|A| = (A^2)^{1/2}$, we have $\|AB\| = \||A|B\| = \|B|A|\| = \||B||A|\|$, and $|A^k| = |A|^k$; so the Cordes inequality with $t = 1/k$ yields

$$(1.1) \quad \|(AB)^k\| \leq \|AB\|^k \leq \|A^k B^k\| \quad (A = A^T, B = B^T).$$

But merely symmetry does not suffice to prove the corresponding spectral-radius inequality. For instance, choosing $A = \mathbf{1}\mathbf{1}^T$ and $B = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$, a quick calculation shows that $\rho(AB)^3 = 1$ while $\rho(A^3 B^3) = 0$. Habibzadeh’s conjecture asks whether requiring nonnegativity in addition suffices. We call a real matrix or vector M nonnegative, and write $M \geq 0$, if every entry of M is nonnegative.

The first main result of this paper resolves the conjecture via a more general inequality.

Theorem 1.1. *Let A and B be nonnegative symmetric matrices. Then, for all integers $k, m \geq 1$,*

$$(1.2) \quad \|(AB)^m\|^k \leq \|(A^k B^k)^m\|,$$

and in particular $\rho(AB)^k \leq \rho(A^k B^k)$ for every integer $k \geq 1$.

For even $k = 2r$, Cordes’ inequality combined with $\|AB\|^2 = \rho(A^2 B^2)$ gives $\|AB\|^{2r} \leq \rho(A^{2r} B^{2r})$ (Lemma 3.3). Moreover, for even k , Theorem 1.1 may omit the nonnegativity assumption.

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For odd $k = 2r + 1$, we deduce Theorem 1.1 from Theorem 1.2 (our second main result) that proves a “Blakley–Roy”-type inequality for a product of arbitrary symmetric nonnegative matrices.

Theorem 1.2 (Path inequality). *Let $k = 2r + 1$ be a positive odd integer, let $t \geq 1$, and let $C_1, \dots, C_t$ be nonnegative symmetric matrices of the same order. For nonnegative vectors u, v ,*

$$(1.3) \quad (u^T C_1 C_2 \cdots C_t v)^k \leq (u^T u)^r (v^T v)^r u^T C_1^k C_2^k \cdots C_t^k v.$$

Our proof of Theorem 1.2 is an entropy argument built on the Gibbs variational principle: the two matrix products in (1.3) are partition functions of weighted paths, and we compare them through a trial law that traverses each edge k times, back and forth (Section 2).

More results. Theorem 1.2 also gives the many-matrix form of the spectral conjecture for odd k , $\rho(C_1 \cdots C_t)^k \leq \rho(C_1^k \cdots C_t^k)$ (Corollary 3.2). For even k and $t \geq 3$ this fails (Remark 3.4). The entropy argument leaves an exact remainder that can characterize equality: for strictly positive A and B and odd $k \geq 3$, equality in (1.2) holds exactly when A and B share a Perron eigenvector (Corollary 4.3), and likewise for the product inequality (Section 4.1). Section 4.2 and Appendix A record an extension of our path based arguments to trees, and other minor generalizations.

2. PROOF OF THE PATH INEQUALITY THEOREM 1.2

For $t = 1$ and $u = v$, inequality (1.3) is the Blakley–Roy inequality $(u^T C u)^k \leq (u^T u)^{k-1} u^T C^k u$, and for $t = 1$ and $u \neq v$ it is the inequality $(u^T C v)^k \leq (u^T u)^r (v^T v)^r u^T C^k v$ of Pate (2012). For one symmetric nonnegative matrix C and unit vectors u, v , Sağlam (2018, Theorem I.3) proves $(u^T C^\ell v)^k \leq (u^T C^k v)^\ell$ for positive integers $k \geq \ell$ of the same parity, which for $\ell = 1$ is Pate’s inequality. Our comparison permits a different matrix on each edge of the path.

In (1.3), $u^T C_1 \cdots C_t v$ and $u^T C_1^k \cdots C_t^k v$ are partition functions of (short) t -edge paths and (long) tk -edge paths. The Gibbs variational principle lower bounds log-partition functions by the energy plus the entropy of any law on its paths. The proof therefore comes down to choosing a law on the long paths whose energy and entropy we can compute. We build this law by gluing together the vertex and edge marginals of the Gibbs law of the short paths. The entropy identities that make the computation possible are those behind the proofs of Sidorenko’s inequality for trees and some extensions (Li and Szegedy, 2025; Szegedy, 2014; Conlon et al., 2018); see Zhao (2023, Chapter 5).

First, we recall Gibbs variational identity, wherein $D(\pi \parallel \sigma)$ denotes relative entropy as usual.

Lemma 2.1 (Gibbs variational identity). *Let $w : \Omega \rightarrow (0, \infty)$ on a finite nonempty set, let $Z = \sum_{x \in \Omega} w(x)$, and let $\mu = w/Z$. For every probability vector π on Ω ,*

$$\log Z = \mathbb{E}_\pi \log w + H(\pi) + D(\pi \parallel \mu).$$

In particular, $\mathbb{E}_\pi \log w + H(\pi) \leq \log Z$, with equality exactly when $\pi = \mu$.

Next, we recall a key property of entropy for Markov laws; this will prove useful for studying long paths. Its entropy, like its energy, is then determined by its pair laws and its one-point laws.

Lemma 2.2 (Entropy of a Markov chain). *Let $Y_0, Y_1, \dots, Y_N$ be a Markov chain on a finite state space. Let η_j be the law of Y_j , and let γ_j be the joint law of (Y_{j-1}, Y_j) . Then*

$$H(Y_0, Y_1, \dots, Y_N) = \sum_{j=1}^N H(\gamma_j) - \sum_{j=1}^{N-1} H(\eta_j).$$

Proof of Theorem 1.2. For $k = 1$ the two sides of (1.3) coincide. Let $k = 2r + 1 \geq 3$, and assume first that all entries of u, v , and the C_i are positive; the nonnegative case follows by continuity.

What we must show. Set $Z = u^T C_1 C_2 \cdots C_t v$, $Q = u^T C_1^k C_2^k \cdots C_t^k v$, $U = u^T u$, and $V = v^T v$; then (1.3) reads $\log Q \geq k \log Z - r \log U - r \log V$. Lemma 2.1 lower bounds $\log Q$ by the energy plus the entropy of any law on long paths. The proof has four steps: (i) write $\log Z$ through the vertex and edge marginals of the Gibbs law of the short paths; (ii) glue from these marginals a

Markov law ν on the long paths; (iii) compute the energy and entropy of ν , which are k times those of the short paths up to some endpoint terms; (iv) show that U^r and V^r absorb the endpoint terms.

(i) Short paths. Write $C(x, y)$ for the (x, y) entry of a matrix C . Then

$$Z = \sum_{x_0, \dots, x_t} w(x), \quad w(x) = u_{x_0} \prod_{i=1}^t C_i(x_{i-1}, x_i) v_{x_t},$$

and $\mu = w/Z$ is a probability law on the paths $x = (x_0, \dots, x_t)$. Let $X = (X_0, \dots, X_t)$ have law μ , let α_i be the law of X_i , and let β_i be the law of (X_{i-1}, X_i) . Since w factorizes along the path, X is a Markov chain, so the equality case of Lemma 2.1 and Lemma 2.2 give

$$(2.1) \quad \log Z = \mathbb{E}_{\alpha_0} \log u + \sum_{i=1}^t \mathbb{E}_{\beta_i} \log C_i + \mathbb{E}_{\alpha_t} \log v + H(\mu), \quad H(\mu) = \sum_{i=1}^t H(\beta_i) - \sum_{i=1}^{t-1} H(\alpha_i).$$

(ii) Long paths and the glued law. Likewise $Q = \sum_y W(y)$ over $y = (y_0, \dots, y_{tk})$, with weight

$$W(y) = u_{y_0} \prod_{i=1}^t \prod_{j=1}^k C_i(y_{(i-1)k+j-1}, y_{(i-1)k+j}) v_{y_{tk}} :$$

edge i of the short path is traversed k times. Let $\mu_k = W/Q$ be their Gibbs law. By Lemma 2.1, $\log Q = \mathbb{E}_\nu \log W + H(\nu) + D(\nu \| \mu_k)$ for every law ν on these paths, and it remains to choose ν . We want $\mathbb{E}_\nu \log W$ and $H(\nu)$ expressed through the α_i and β_i of (2.1). The energy is k times that of the short paths if every step across edge i has pair law β_i ; a step with pair law β_i leads from α_{i-1} to α_i , so to cross the edge again the chain has to turn back. Let $Y = (Y_0, \dots, Y_{tk})$ be the Markov chain with $Y_0 \sim \alpha_0$ whose steps in block i , the steps $(i-1)k+1, \dots, ik$, alternate between the forward and the backward transitions

$$(2.2) \quad P_i^+(y' | y) = \frac{\beta_i(y, y')}{\alpha_{i-1}(y)}, \quad P_i^-(y | y') = \frac{\beta_i(y, y')}{\alpha_i(y')},$$

beginning with P_i^+ , and let ν be its law. These are the forward and the reversed transitions of the conditioned walk in Sağlam (2018, Section 3); for one matrix, his proof lengthens the walk by two steps, with one reversed transition at a uniformly random position, and iterates, whereas we insert all $2r$ extra steps of each block at once, which is what lets the matrix change from edge to edge. Since β_i has marginals α_{i-1} and α_i , a forward step carries α_{i-1} to α_i with pair law β_i , and a backward step carries α_i to α_{i-1} with pair law β_i^T . Oddness of k enters here: block i starts from α_{i-1} and ends at α_i , where block $i+1$ begins, so the blocks fit together, and every block has $r+1$ forward and r backward steps.

(iii) Energy and entropy of the glued law. Symmetry enters next: as $C_i = C_i^T$, a backward traversal of edge i has the same expected energy as a forward one, $\mathbb{E}_{\beta_i^T} \log C_i = \mathbb{E}_{\beta_i} \log C_i$, so

$$(2.3) \quad \mathbb{E}_\nu \log W = \mathbb{E}_{\alpha_0} \log u + k \sum_{i=1}^t \mathbb{E}_{\beta_i} \log C_i + \mathbb{E}_{\alpha_t} \log v.$$

For $H(\nu)$, Lemma 2.2 needs the laws of the tk pairs and of the $tk-1$ interior states. The pairs contribute $k \sum_{i=1}^t H(\beta_i)$. Inside block i , the $2r$ interior states have law α_i after a forward step and α_{i-1} after a backward step, r times each; the $t-1$ states where consecutive blocks meet have laws $\alpha_1, \dots, \alpha_{t-1}$. Hence α_0, α_t occur r times each, and each α_i with $0 < i < t$ occurs $2r+1 = k$ times:

$$(2.4) \quad H(\nu) = k \sum_{i=1}^t H(\beta_i) - k \sum_{i=1}^{t-1} H(\alpha_i) - rH(\alpha_0) - rH(\alpha_t) = kH(\mu) - rH(\alpha_0) - rH(\alpha_t).$$

(iv) Endpoints. Subtracting k times (2.1) from the sum of (2.3) and (2.4), the edge terms cancel and only the endpoints remain:

$$\log Q - k \log Z = D(\nu \| \mu_k) - r(2\mathbb{E}_{\alpha_0} \log u + H(\alpha_0)) - r(2\mathbb{E}_{\alpha_t} \log v + H(\alpha_t)).$$

Put $p_u(a) = u_a^2/U$. Lemma 2.1 with $w(a) = u_a^2$, whose Gibbs law is p_u , gives $2\mathbb{E}_{\alpha_0} \log u + H(\alpha_0) = \log U - D(\alpha_0 \| p_u)$; likewise $2\mathbb{E}_{\alpha_t} \log v + H(\alpha_t) = \log V - D(\alpha_t \| p_v)$ with $p_v(b) = v_b^2/V$. Without the remainders, this endpoint bound is Sağlam (2018, Lemma 3.6). Hence

$$(2.5) \quad \log \frac{QU^r V^r}{Z^k} = D(\nu \| \mu_k) + rD(\alpha_0 \| p_u) + rD(\alpha_t \| p_v).$$

The right side is nonnegative, which is (1.3) for positive data. For nonnegative data, apply (1.3) to $u + \varepsilon \mathbf{1}$, $v + \varepsilon \mathbf{1}$, and $C_i + \varepsilon \mathbf{1}\mathbf{1}^T$, and let $\varepsilon \downarrow 0$. $\square$

Remark 2.3. Symmetry enters the proof only through $\mathbb{E}_{\beta_i^T} \log C_i = \mathbb{E}_{\beta_i} \log C_i$. Without it, a backward traversal of edge i carries the weight C_i^T , and the same proof, including (2.5), gives, for nonnegative matrices $C_1, \dots, C_t$ of any compatible shapes and nonnegative endpoint vectors,

$$(2.6) \quad (u^T C_1 \cdots C_t v)^k \leq (u^T u)^r (v^T v)^r u^T F_k(C_1) \cdots F_k(C_t) v, \quad F_k(C) = (CC^T)^r C,$$

with $Q = u^T F_k(C_1) \cdots F_k(C_t) v$ in (2.5). For $t = 1$ this is Pate's inequality (Pate, 2012, Theorem 4).

3. PROOF OF THEOREM 1.1 AND RELATED NORM INEQUALITIES

Nonnegative maximizing singular vectors turn the endpoint comparison into a norm comparison. Repeating the factors then gives the spectral radius and the odd part of Theorem 1.1.

Corollary 3.1. Let k be a positive odd integer, and let $C_1, \dots, C_t$ be real symmetric matrices with nonnegative entries. Then

$$(3.1) \quad \|C_1 C_2 \cdots C_t\|^k \leq \|C_1^k C_2^k \cdots C_t^k\|.$$

Proof. Set $D = C_1 C_2 \cdots C_t$ and $E = C_1^k C_2^k \cdots C_t^k$, both entrywise nonnegative, and assume $D \neq 0$. Since $D^T D \geq 0$, every real vector y satisfies $|y|^T D^T D |y| \geq y^T D^T D y$, so $D^T D$ has a unit eigenvector $y \geq 0$ for its largest eigenvalue $\|D\|^2$. Set $x = Dy/\|D\|$; this is a unit vector with $x \geq 0$ and $x^T Dy = \|D\|$. Theorem 1.2 gives

$$\|D\|^k = (x^T Dy)^k \leq x^T E y \leq \|E\|. \quad \square$$

Corollary 3.2. Let k be a positive odd integer, and let $C_1, \dots, C_t$ be real symmetric matrices with nonnegative entries. Then $\rho(C_1 C_2 \cdots C_t)^k \leq \rho(C_1^k C_2^k \cdots C_t^k)$.

Proof. Applied to the tm factors of $(C_1 \cdots C_t)^m$, inequality (3.1) gives

$$(3.2) \quad \|(C_1 \cdots C_t)^m\|^k \leq \|(C_1^k \cdots C_t^k)^m\|.$$

Apply Gelfand's formula $\rho(X) = \lim_{m \rightarrow \infty} \|X^m\|^{1/m}$ to conclude $\rho(C_1 \cdots C_t)^k \leq \rho(C_1^k \cdots C_t^k)$. $\square$

3.1. Two matrices and all exponents. For two factors, the even exponents can be handled without the entropy argument. The following consequence of Cordes' inequality isolates where entrywise nonnegativity is needed.

Lemma 3.3. Let $A = A^T$ and $B = B^T$. Then $\|AB\|^2 = \rho(A^2 B^2)$, and, for all integers $q, m \geq 1$,

$$(3.3) \quad \|(AB)^m\|^{2q} \leq \rho(A^{2q} B^{2q})^m \leq \|(A^{2q} B^{2q})^m\|.$$

Proof. Since XY and YX have the same nonzero eigenvalues,

$$\|AB\|^2 = \rho((AB)^T (AB)) = \rho(BA^2 B) = \rho(A^2 B^2).$$

Apply this identity to the symmetric matrices A^q, B^q and use (1.1) to obtain

$$\|AB\|^{2q} \leq \|A^q B^q\|^2 = \rho(A^{2q} B^{2q}).$$

The first inequality in (3.3) is submultiplicativity; the second follows from $\rho(M^m) \leq \|M^m\|$. $\square$

Proof of Theorem 1.1. For odd k , apply Corollary 3.1 to the $2m$ alternating factors $A, B, \dots, A, B$ of $(AB)^m$. For even $k = 2q$, apply Lemma 3.3. This proves (1.2). Taking m -th roots in (1.2), Gelfand's formula gives

$$\rho(AB)^k = \lim_{m \rightarrow \infty} \|(AB)^m\|^{k/m} \leq \lim_{m \rightarrow \infty} \|(A^k B^k)^m\|^{1/m} = \rho(A^k B^k). \quad \square$$

3.2. Limits of the hypotheses. The two-matrix conclusion uses fewer hypotheses at even exponents than the many-factor conclusion, and the next example separates the two. (Positive semidefiniteness does not replace entrywise nonnegativity in the many-factor statements either: for three positive definite 2×2 factors, both (3.1) and (1.3) fail, as an easy example shows. Comparisons of three or more positive semidefinite matrices require the multivariate trace inequalities of Sutter et al. (2017); see Hiai et al. (2017) for their log-majorization form.)

Remark 3.4. Corollaries 3.1 and 3.2 fail for even k . Let E_{ij} be the 3×3 matrix with a single 1, in position (i, j) , and let $C_1 = E_{23} + E_{32}$, $C_2 = E_{13} + E_{31}$, and $C_3 = E_{12} + E_{21}$. Then

$$C_1 C_2 C_3 = E_{22}, \quad C_1^2 C_2^2 C_3^2 = (E_{22} + E_{33})(E_{11} + E_{33})(E_{11} + E_{22}) = 0,$$

so $\|C_1 C_2 C_3\| = \rho(C_1 C_2 C_3) = 1$ while $C_1^2 C_2^2 C_3^2 = 0$.

4. EQUALITY CASES AND EXTENSIONS

The identity (2.5) writes the logarithmic gap in (1.3) as a sum of three relative entropies. Equality therefore forces the glued law to coincide with the Gibbs law of the long paths, and a message calculation turns that coincidence into a condition on the matrices.

4.1. Equality. Let $k = 2r + 1 \geq 3$, let $C_1, \dots, C_t$ be strictly positive matrices of compatible rectangular shapes, and let the endpoint vectors u, v be strictly positive. Let Z, Q, U, V , the laws $\mu, \alpha_i, \beta_i, \nu, \mu_k$, and the endpoint laws p_u, p_v be as in the proof of Theorem 1.2 and Remark 2.3, so that

$$(4.1) \quad \Delta := \log \frac{QU^r V^r}{Z^k} = D(\nu \| \mu_k) + rD(\alpha_0 \| p_u) + rD(\alpha_t \| p_v).$$

Proposition 4.1. With the positive data above, equality in (2.6) holds if and only if there are positive unit vectors $a_0, \dots, a_t$ and positive numbers $\sigma_1, \dots, \sigma_t$ such that

$$(4.2) \quad C_i a_i = \sigma_i a_{i-1}, \quad C_i^T a_{i-1} = \sigma_i a_i \quad (1 \leq i \leq t), \quad u \parallel a_0, \quad v \parallel a_t.$$

For strictly positive symmetric C_i , this is equivalent to a common positive Perron eigenvector proportional to both u and v .

Proof. Define the left and right messages

$$l_0 = u, \quad l_i = C_i^T \cdots C_1^T u, \quad h_t = v, \quad h_i = C_{i+1} \cdots C_t v.$$

Summing the weights on either side of a vertex or edge gives

$$(4.3) \quad \alpha_i(a) = \frac{l_i(a)h_i(a)}{Z}, \quad \beta_i(a, b) = \frac{l_{i-1}(a)C_i(a, b)h_i(b)}{Z}.$$

Thus the transitions in (2.2) are $C_i(a, b)h_i(b)/h_{i-1}(a)$ and $C_i(a, b)l_{i-1}(a)/l_i(b)$. Write $y_{i,j} = y_{(i-1)k+j}$ for coordinate j of block i , so that $y_{i,k} = y_{i+1,0}$. Multiplying these transitions and cancelling the edge weights against those in μ_k gives

$$(4.4) \quad \frac{\nu(y)}{\mu_k(y)} = \frac{Q}{Z} \prod_{i=1}^t \prod_{a=1}^r \frac{h_i(y_{i,2a-1})}{l_i(y_{i,2a-1})} \frac{l_{i-1}(y_{i,2a})}{h_{i-1}(y_{i,2a})}.$$

All interior coordinates can be varied independently and all configurations have positive probability. Since $r \geq 1$, each level $i = 0, \dots, t$ appears in at least one factor of this product. Hence the ratio in

(4.4) is constant if and only if $l_i \parallel h_i$ at every level. A constant ratio is one because both laws are normalized.

If equality holds in (2.6), then $\Delta = 0$ and (4.1) gives $\nu = \mu_k$. The proportionalities just obtained follow. Conversely, those proportionalities give $\nu = \mu_k$, and the endpoint cases of (4.3) give $\alpha_0 = p_u$ and $\alpha_t = p_v$. Thus all three terms of (4.1) vanish.

Normalize the proportional messages to unit vectors a_i . The recursions $h_{i-1} = C_i h_i$ and $l_i = C_i^T l_{i-1}$ give $C_i a_i = s_i a_{i-1}$ and $C_i^T a_{i-1} = t_i a_i$ for positive scalars. Taking the inner product with a_{i-1} and a_i , respectively, gives $s_i = t_i$, proving (4.2). Conversely, (4.2) and the endpoint proportionalities imply $l_i \parallel h_i$ by recursion.

If C_i is symmetric and strictly positive, both a_{i-1} and a_i are positive eigenvectors of C_i^2 with eigenvalue σ_i^2 . The Perron eigenspace of the strictly positive matrix C_i^2 is one-dimensional, so the unit vectors coincide. Equation (4.2) then makes this common vector a Perron eigenvector of every C_i . The converse follows by substituting the common eigenvector into (4.2). $\square$

Corollary 4.2. *For strictly positive symmetric $C_1, \dots, C_t$ and odd $k \geq 3$, equality in (3.1) holds if and only if the factors share a positive Perron eigenvector.*

Proof. The product $D = C_1 \cdots C_t$ has strictly positive left and right unit singular vectors x, y for its largest singular value. If equality holds in (3.1), both inequalities $\|D\|^k = (x^T D y)^k \leq x^T E y \leq \|E\|$, with $E = C_1^k \cdots C_t^k$, are equalities. Proposition 4.1 gives a common Perron vector. Conversely, if $C_i a = \lambda_i a$ for one positive unit vector a , symmetry gives $\|C_i\| = \lambda_i$. The eigenvector lower bound and submultiplicativity show $\|D\| = \prod_i \lambda_i$ and $\|E\| = \prod_i \lambda_i^k$. $\square$

Corollary 4.3. *Let A and B be strictly positive symmetric matrices, let $k \geq 3$ be odd, and let $m \geq 1$. Equality holds in (1.2) if and only if A and B share a positive Perron eigenvector.*

Proof. For odd k , inequality (1.2) is (3.1) for the $2m$ alternating factors $A, B, \dots, A, B$, so Corollary 4.2 applies. $\square$

4.2. Extensions. The proof of Theorem 1.2 uses the path only through Lemma 2.2, which is the path case of the entropy formula for laws that factorize along a tree. Appendix A proves the tree version: for a finite tree T with a nonnegative rectangular matrix C_e on each edge and a nonnegative vector u_ℓ at each leaf ℓ , the partition function Z_T of (A.3) satisfies

$$Z_T(C; u)^k \leq \left(\prod_\ell (u_\ell^T u_\ell)^r \right) Z_T(F_k(C); u),$$

where $F_k(C)$ replaces each edge matrix by $(C_e C_e^T)^r C_e$ (Theorem A.2). Its exact remainder is (2.5) with one relative entropy per leaf and an additional term $r \sum_v (d_v - 2) H(\alpha_v)$ over the internal vertices v , of degree d_v . The leaf factors are not a homomorphism-density normalization; under that normalization the branching term in (A.5) changes sign and the comparison can fail.

The remainder also bounds the distance to equality. By Pinsker's inequality, with Δ as in (4.1),

$$\|\nu - \mu_k\|_1 \leq \sqrt{2\Delta}, \quad \|\alpha_0 - p_u\|_1, \|\alpha_t - p_v\|_1 \leq \sqrt{2\Delta/r}.$$

These estimates compare laws; they do not say that the matrices are close to a family with a common eigenvector.

The even case of Theorem 1.1 extends in a different direction, to all unitarily invariant norms; the odd case does not.

Remark 4.4. *For even exponents, Lemma 3.3 holds in every unitarily invariant norm $\|\cdot\|$: for symmetric A, B and integers $q, m \geq 1$,*

$$(4.5) \quad \|(AB)^m\|^{2q} \leq \|(A^{2q} B^{2q})^m\|, \quad |X| = (X^T X)^{1/2}.$$

Indeed, the j -th compound matrices of A and B are symmetric, and Lemma 3.3 applied to them gives $\prod_{i \leq j} \sigma_i((AB)^m)^{2q} \leq \prod_{i \leq j} \sigma_i((A^{2q} B^{2q})^m)$ for every j , where $\sigma_1(X) \geq \sigma_2(X) \geq \dots$ are the singular

values; this weak log-majorization implies (4.5) (Hiai and Petz, 2014, Theorems 6.5 and 6.23). The power must stand inside the norm: for $A = B = I_n$ the trace norms give $n^{2a} > n$. For odd k under entrywise nonnegativity there is no such extension, since the compound matrices of a nonnegative matrix need not be nonnegative. With $J = \mathbf{1}\mathbf{1}^T$ and $Q = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$ from Section 1, write $Q = Q_+ - Q_-$ with $Q_{\pm} \geq 0$, and let $A = J \oplus J$ and $B = \begin{pmatrix} Q_+ & Q_- \\ Q_- & Q_+ \end{pmatrix}$, both symmetric and nonnegative. Conjugation by $\frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ I & -I \end{pmatrix}$ fixes A and takes B to $(Q_+ + Q_-) \oplus Q$. Since $(JQ)^2 = JQ$ has rank one and $(J^3Q^3)^2 = 0$, the matrix $(AB)^2$ has rank two while $(A^3B^3)^2$ has rank one, so the cubed singular values of $(AB)^2$ are not weakly log-majorized by those of $(A^3B^3)^2$.

AI USAGE

The results in this paper were developed through extensive use of AI assistants: GPT Pro for the main line of argument, and Claude for the writing, literature search, and some examples. None of the claims were obtained zero-shot: they emerged through sustained back-and-forth discussion, including failed attempts, counterexamples, audits, and repairs. The main idea to pursue the norm inequality was introduced by the author.

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APPENDIX A. TREES

Lemma 2.2 is the path case of the gluing lemma below, and the proof of Theorem 1.2 is the path case of the proof of Theorem A.2. The vertex and edge marginals of a Gibbs law on a tree can be glued without a compatibility condition around cycles, and the glued law has an explicit entropy.

Lemma A.1 (Gluing on a tree). *Let $T = (V, E)$ be a finite tree, with a finite nonempty state space Ω_v at each vertex. Suppose α_v is a strictly positive probability vector on Ω_v , and for each edge $\{v, w\}$ the strictly positive pair law β_{vw} has marginals α_v, α_w , with $\beta_{vw} = \beta_{vw}^T$. There is a probability law η with these marginals given by*

$$(A.1) \quad \eta(x) = \frac{\prod_{\{v,w\} \in E} \beta_{vw}(x_v, x_w)}{\prod_{v \in V} \alpha_v(x_v)^{d_v-1}}, \quad d_v = \deg_T(v).$$

It satisfies

$$(A.2) \quad H(\eta) = \sum_{\{v,w\} \in E} H(\beta_{vw}) - \sum_{v \in V} (d_v - 1)H(\alpha_v).$$

Every strictly positive law that factorizes into vertex and edge weights on T has the form (A.1) for its own marginals.

Proof. Choose a root o . Sample its state with law α_o , and, independently along the branches conditional on their parent states, sample each child w of v with transition $\beta_{vw}(x_v, x_w)/\alpha_v(x_v)$. Each transition sums to one, and induction from the root verifies the prescribed vertex and edge marginals. Multiplying the transitions and $\alpha_o(x_o)$ gives (A.1); in particular, the law is independent of the root. Taking the expectation of $-\log \eta$ proves (A.2).

For a law that factorizes into positive vertex and edge weights, conditioning on the state at a vertex separates the components left after deleting that vertex: their conditional weights are products over disjoint variables. Thus each child subtree is conditionally independent of the other subtrees and of the ancestors, given its parent state. Factoring the law from the root therefore gives the same transitions and (A.1). $\square$

A.1. Partition functions and their odd expansions. Let $T = (V, E)$ have at least one edge, let $L = \{v : d_v = 1\}$ be its leaves, and assign a finite nonempty state space Ω_v to each vertex. Choose an orientation for each edge $e = (v, w)$ and a nonnegative matrix C_e of shape $|\Omega_v| \times |\Omega_w|$. At each leaf choose a nonnegative vector u_ℓ indexed by Ω_ℓ , and put $U_\ell = u_\ell^T u_\ell$. There are no additional

vertex weights at the internal vertices. Define

$$(A.3) \quad Z_T(C; u) = \sum_{x \in \prod_{v \in V} \Omega_v} \prod_{e=(v,w) \in E} C_e(x_v, x_w) \prod_{\ell \in L} u_\ell(x_\ell).$$

All sums use counting measure. Reversing an edge orientation transposes its matrix and leaves this partition function unchanged.

For a positive odd integer $k = 2r + 1$, let $F_k(C) = (CC^T)^r C$ as in Remark 2.3. Replace each edge $e = (v, w)$ by a path of length k . Its successive state spaces alternate between copies of Ω_v and Ω_w , and its matrices alternate between C_e and C_e^T . The $2r$ inserted vertices have new coordinates, even when their state spaces are copies of the same Ω_v . Keep the original leaf weights. Summing over the inserted coordinates contracts this path to $F_k(C_e)$, so the expanded tree has partition function

$$Q = Z_T(F_k(C); u), \quad F_k(C) = (F_k(C_e))_{e \in E}.$$

The identity $F_k(C^T) = F_k(C)^T$ makes the expansion independent of the chosen edge orientations.

For strictly positive data, let μ be the Gibbs law of $Z = Z_T(C; u)$, with vertex marginals α_v and edge marginals β_e . Let μ_k be the Gibbs law on the expanded tree. Give each copy of Ω_v the marginal α_v , and give the successive edges in the expansion of e pair laws $\beta_e, \beta_e^T, \dots, \beta_e$. These marginals agree at every shared vertex. Lemma A.1 defines a law ν on the expanded tree from them. Finally, set

$$p_\ell(a) = \frac{u_\ell(a)^2}{U_\ell}.$$

The entropy comparison distinguishes the leaves, where a boundary factor is needed, from the branching vertices, which give a nonnegative remainder.

Theorem A.2. *For the tree and nonnegative data in (A.3), and every odd integer $k = 2r + 1 \geq 3$,*

$$(A.4) \quad Z_T(C; u)^k \leq \left(\prod_{\ell \in L} U_\ell^r \right) Z_T(F_k(C); u).$$

If all entries of the edge matrices and leaf vectors are strictly positive, the laws defined above satisfy the exact identity

$$(A.5) \quad \log \frac{Q \prod_{\ell \in L} U_\ell^r}{Z^k} = D(\nu \| \mu_k) + r \sum_{\ell \in L} D(\alpha_\ell \| p_\ell) + r \sum_{v \in V \setminus L} (d_v - 2) H(\alpha_v).$$

Proof. First assume strict positivity. Denote the weights of the original and expanded configurations by w and W , respectively. By the Gibbs identity,

$$(A.6) \quad \log Z = \sum_{e \in E} \mathbb{E}_{\beta_e} \log C_e + \sum_{\ell \in L} \mathbb{E}_{\alpha_\ell} \log u_\ell + H(\mu).$$

Each forward traversal of edge e has expected log weight $\mathbb{E}_{\beta_e} \log C_e$; a backward traversal has the same expectation because both the pair law and the matrix are transposed. The original leaves retain their marginals. Thus

$$(A.7) \quad \mathbb{E}_\nu \log W = k \sum_{e \in E} \mathbb{E}_{\beta_e} \log C_e + \sum_{\ell \in L} \mathbb{E}_{\alpha_\ell} \log u_\ell.$$

To compute $H(\nu)$, apply Lemma A.1 to both trees. The expanded edges contribute $k \sum_e H(\beta_e)$. Each original vertex retains degree d_v and marginal α_v , and each incident edge introduces r degree-two vertices with that marginal. Consequently,

$$(A.8) \quad \begin{aligned} H(\nu) &= k \sum_e H(\beta_e) - \sum_v (d_v - 1) H(\alpha_v) - r \sum_v d_v H(\alpha_v) \\ &= k H(\mu) + r \sum_v (d_v - 2) H(\alpha_v). \end{aligned}$$

The second equality uses $k - 1 = 2r$.

Apply Lemma 2.1 to W with trial law ν , and subtract k times (A.6). Equations (A.7) and (A.8) give

$$\begin{aligned} \log Q - k \log Z &= D(\nu \| \mu_k) + r \sum_{v \notin L} (d_v - 2) H(\alpha_v) \\ &\quad - r \sum_{\ell \in L} (H(\alpha_\ell) + 2\mathbb{E}_{\alpha_\ell} \log u_\ell). \end{aligned}$$

For each leaf, the definition of p_ℓ gives

$$H(\alpha_\ell) + 2\mathbb{E}_{\alpha_\ell} \log u_\ell = \log U_\ell - D(\alpha_\ell \| p_\ell).$$

Substitution proves (A.5). Its right side is nonnegative, since $d_v \geq 2$ at every internal vertex, proving (A.4) for positive data. Adding $\varepsilon > 0$ to every matrix and vector entry and letting $\varepsilon \downarrow 0$ proves (A.4) for nonnegative data. $\square$

For the path with vertices $0, \dots, t$, the matrix C_i on edge $(i - 1, i)$, and leaf vectors u, v , inequality (A.4) is (2.6), and (A.5) is (2.5), since every internal vertex has degree two. For strictly positive data and $k \geq 3$, a vertex of degree at least three with at least two states makes (A.4) strict, by its term in (A.5).